Linear Algebra for Computer Science

Lecture 24

Complex Matrices, Eigenpairs of Real Symmetric and Conjugate Symmetric Matrices

Complex Numbers

$A \in \mathbb{R}^{m\times n} \longrightarrow A \in \mathbb{C}^{m\times n}$ e.g 3 I

$u \in \mathbb{R}^{n} \longrightarrow u \in \mathbb{C}^{n}$

$c = a + bi \quad c \in \mathbb{C}$

$a, b \in \mathbb{R}$

$$u = \begin{bmatrix} 4 \\ 3i-2 \\ 6i \end{bmatrix} \in \mathbb{C}^3$$

$$\bar{c} = \overline{a+bi} = a - bi$$

conjugate مزدوج

$$\bar{u} = \begin{bmatrix} 4 \\ -3i-2 \\ -6i \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3+4i \\ i & 4 \\ 2-i & 1+i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 2 & 3-4i \\ -i & 4 \\ 2+i & 1-i \end{bmatrix}$$

$$\overline{AB} = \bar{A}\,\bar{B} \qquad \cancel{A \in \mathbb{R}^{m\times n}}\ A \in \mathbb{C}^{m\times n} \quad B \in \mathbb{C}^{n\times p}$$

Length, orthogonality, and dot product

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بردار

- اندازه $\|v\|$
- تعامد $v \perp u$

$$u, v \in \mathbb{R} \quad \begin{cases} \|v\|^2 = v^T v = \langle v, v \rangle \\ u \perp v \Rightarrow \langle u, v \rangle = 0 \Rightarrow u^T v = 0 \end{cases}$$

Length of Complex Numbers

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for complex vectors

$$V = \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad V^T V = \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = i^2 + 1 = -1 + 1 = 0$$

تعجب! کرده

$$V^T V = \sum V_i^2$$

$$\|V\|^2 = \sum |V_i|^2 = \sum V_i \bar{V}_i$$

$$\left\| \begin{bmatrix} i \\ 1 \end{bmatrix} \right\|^2 = |i|^2 + |1| = 1 + 1 = 2$$ makes sense

$V \in \mathbb{C}^n$

$$\|V\|^2 = \sum_{i=1}^{n} |V_i|^2 = \sum_{i=1}^{n} V_i \bar{V}_i = \begin{bmatrix} \bar{V}_1 & \bar{V}_2 & \cdots & \bar{V}_n \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} V_1 & \text{—} & V_n \end{bmatrix} \begin{bmatrix} \bar{V}_1 \\ | \\ \bar{V}_n \end{bmatrix}$$

$$= \bar{V}^T V = V^T \bar{V}$$

Inner product for complex numbers

for complex vectors u, v $\quad \langle u, v \rangle = \bar{v}^T u$

$$\|u\|^2 = \bar{u}^T u = \langle u, u \rangle$$

$$u \perp v \iff \langle u, v \rangle = 0 \iff \boxed{\bar{v}^T u = 0}$$

Length and Orthogonality for complex numbers

$$\langle v,u\rangle = \underbrace{\bar{u}^T v}_{1\times 1} = v^T\bar{u} = \overline{\bar{v}^T u} = \overline{\langle u,v\rangle}$$ conjugate symmetry

$$u,v\in\mathbb{C}^n \qquad u\perp v \Leftrightarrow \bar{v}^T u = 0$$

$$\forall u\in\mathbb{C}^n \qquad \|u\| = \sqrt{\langle u,u\rangle} = \sqrt{\bar{u}^T u}$$

$\bar{u}^T = u^* = u^H$ Conjugate transpose

= Hermitian transpose

Orthogonal Matrix => Unitary Matrix

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$U \in \mathbb{C}^{n\times n}$ is an orthogonal matrix

$\Rightarrow U = [u_1 \; u_2 \cdots u_n]$ has orthonormal columns (rows)

$$\left.\begin{array}{ll} u_i \perp u_j & i \neq j \\ \|u_i\| = 1 & \end{array}\right\} \Rightarrow \langle u_i, u_j \rangle = \bar{u}_j^T u_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\begin{bmatrix} — & \bar{u}_1^T & — \\ — & \bar{u}_2^T & — \\ & \vdots & \\ — & \bar{u}_n^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = I$$

$$\boxed{\bar{U}^T U = I}$$

Conjugate Transpose (Hermitian Transpose)

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$$(\bar{U})^T = \bar{U}^T = U^* = U^H$$

$$A \in \mathbb{C}^{m\times n} \qquad (\bar{A})^T = \bar{A}^T = A^* = A^H = A'$$

conjugate transpose of A

Hermitian transpose

MATLAB

Orthogonal matrices => Unitary Matrices

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$$UU^T = U^T U = I \; / \; UU^H = U^H U = I$$

Hermitian (Conjugate Symmetric) Matrices

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A symmetric $\Longrightarrow A^T = A$

$A \in \cancel{\mathbb{R}}\, \mathbb{C}^{n \times n}$ Conjugate Symmetric $\quad \bar{A}^T = A$

$A^* = A$

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 4 \end{bmatrix} \qquad A^* = \begin{bmatrix} 2 & 1-i \\ 1+i & 4 \end{bmatrix}$$

Egienvalues of Real Symmetric Matrices

$A \in \mathbb{R}^{n\times n}$ is symmetric $A^T = A$ LA24 V

Let (λ, v) be an eigenpair of A.

$\begin{pmatrix}\lambda\in\mathbb{C}\\ v\in\mathbb{C}^n\end{pmatrix}$

$$Av = \lambda v \Rightarrow \bar{v}^T A v = \lambda \bar{v}^T v = \lambda \|v\|^2 \quad \mathrm{I}$$

$$\underbrace{\bar{v}^T A v}_{1\times 1} = (\underbrace{\bar{v}^T A v}_{1\times 1})^T = v^T A^T \bar{v} = v^T A \bar{v}$$

$$\overline{\bar{v}^T A v} = \overline{v^T A \bar{v}} = \bar{v}^T \bar{A} v \underset{A\in\mathbb{R}^{n\times n}}{=} \bar{v}^T A v \Rightarrow \boxed{\bar{v}^T A v \in \mathbb{R}} \quad \mathrm{II}$$

$$\mathrm{I} \Rightarrow \left.\begin{array}{l} \bar{v}^T A v = \lambda \|v\|^2 \\ \bar{v}^T A v \in \mathbb{R} \\ \|v\|^2 \end{array}\right\} \lambda \in \mathbb{R}$$

Egienvalues of Real Symmetric Matrices

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$\Rightarrow$ For any ~~symmetr~~ ~~real~~ real symmetric matrix A $(A \in \mathbb{R}^{n\times n}, A^T = A)$ all eigenvalues are real. (same thing is ~~ture~~ true for Hermitian matrices $A \in \mathbb{C}^{n\times n}$, $A^* = A$)

Egienvalues of Real Symmetric Matrices

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Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix and (v_1, λ_1) & (v_2, λ_2) are eigenpairs with different eigenvalues, $\lambda_1 \neq \lambda_2$

$$A v_1 = \lambda_1 v_1 \Longrightarrow \boxed{\bar{v}_2^T A v_1 = \lambda_1 \bar{v}_2^T v_1} \quad \text{I}$$

$$A v_2 = \lambda_2 v_2 \Longrightarrow \bar{v}_1^T A v_2 = \lambda_2 \bar{v}_1^T v_2$$

$$\lambda_1 \neq \lambda_2$$

$$\Downarrow$$

$$\overline{\bar{v}_1^T A v_2} = \overline{\lambda_2 \bar{v}_1^T v_2}$$

$$\Downarrow$$

real

$$v_1^T A \bar{v}_2 = \bar{\lambda}_2 v_1^T \bar{v}_2$$

$$\bar{\lambda}_2 = \lambda_2 \quad \Downarrow \quad T$$

$$\boxed{\bar{v}_2^T A v_1 = \lambda_2 \bar{v}_2^T v_1} \quad \text{II}$$

$$\left.\begin{array}{l} \text{I}, \text{II} \Longrightarrow \lambda_1 \bar{v}_2^T v_1 = \lambda_2 \bar{v}_2^T v_1 \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \bar{v}_2^T v_1 = 0 \Longrightarrow v_1 \perp v_2$$

Egienvalues of Real Symmetric Matrices

$A \in \mathbb{R}^{n \times n}$ Real symmetric $\Rightarrow$

Eigenvectors correspondig to different eigenvalues are orthogonal.

Real Symmetric Matrices

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eig3

$A \in \mathbb{R}^{n\times n}$ is a real symetric matrix

all the eigenvalues are real.

the eigenvectors are (can be chosen to be) orthogonal.

(Also true for Hermitian matrices $\bar{A}^T = A$)