Linear Algebra for Computer Science

Lecture 27

Singular Value Decomposition

Remember: Diagonalization & Eigendecomposition

LA 27

I

Eigenvalues $Av_i = \lambda_i v_i$ $\quad A \in \mathbb{R}^{n\times n}$, $v_i \in \mathbb{R}^n$

Diagonalization

Eigenbasis $\underline{v_1, v_2, \dots, v_n}$

$AV = V\Lambda$

$A = V\Lambda V^{-1}$ $\quad \Lambda = V^{-1}AV$

$Ax = \underline{V\Lambda \underbrace{V^{-1}x}_{x'}}$

$A \in \mathbb{R}^{n\times n} \rightarrow f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $\quad f(x) = Ax$

$f(x) = Ax$

$\underline{v_1, v_2, \dots, v_n}$ $\quad \underline{v_1, v_2, \dots, v_n}$

only for square matrices
Eigenbasis might be complex
Eigenbasis might not be ortho

Problem with Eigenvalue Decomposition

K. N. Toosi
University of Technology

- Only works for square matrices
- Eigen-basis might not exist
- Eigen-basis might not be orthogonal
- Eigen-basis might be complex for real matrices

Problem with Eigenvalue Decomposition

K. N. Toosi
University of Technology

- Only works for square matrices
- Eigen-basis might not exist
- Eigen-basis might not be orthogonal
- Eigen-basis might be complex for real matrices

Solution: Use different bases for domain and co-domain!

Diagonalization using two different bases

Eigenbasis might not be a good
might not be diagonalizable

$A \in \mathbb{R}^{m\times n}$ $\qquad f: \mathbb{R}^n \to \mathbb{R}^m$

$f(x) = Ax$ $\qquad v_1, v_2, \dots, v_n \in \mathbb{R}^n \qquad u_1, u_2, \dots, u_m \in \mathbb{R}^m$

$\boxed{y = Ax}$ $\qquad V = [v_1, v_2, \dots, v_n] \in \mathbb{R}^{n\times n} \qquad U = [u_1, u_2, \dots, u_m] \in \mathbb{R}^{m\times m}$

x' = representation of x in basis $v_1, v_2, \dots, v_n = V^{-1}x$

y' = " " y in $u_1, u_2, \dots, u_m = U^{-1}y$

$$\Rightarrow \left.\begin{aligned} x &= Vx' \\ y &= Uy' \end{aligned}\right\} \quad Uy' = AVx' \Rightarrow y' = \underbrace{U^{-1}AV}\, x'$$

like it to be diagonal

$$\underset{\in\mathbb{R}^m}{y'} = \underbrace{U^{-1}AV}_{m\times n}\ \underset{\in\mathbb{R}^n}{x'}$$

diagonal

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 & 0 \end{bmatrix}$$

Diagonalization using two different bases

K. N. Toosi
University of Technology

$$\Rightarrow \left.\begin{aligned} x &= Vx' \\ y &= Uy' \end{aligned}\right\} \quad Uy' = AVx' \Rightarrow y' = \underbrace{U^{-1}AV}_{\text{like it to be diagonal}} x'$$

$$\underset{\in \mathbb{R}^m}{y'} = \underbrace{U^{-1}AV}_{m \times n} \underset{\in \mathbb{R}^n}{x'}$$

diagonal

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 & 0 \end{bmatrix}$$

Like $v_1, v_2, \dots, v_n$ be orthonormal, so be $u_1, u_2, \dots, u_m$

$$\Rightarrow U^{-1} = U^T, \quad V^{-1} = V^T$$

$$y' = U^{-1}AVx' \Rightarrow y' = U^TAVx'$$

We like
$$\begin{cases} U^TU = UU^T = I_{m \times m} \\ V^TV = VV^T = I_{n \times n} \\ \Sigma = \underline{U^TAV} \text{ be diagonal} \end{cases}$$

A new decomposition!

II

Eigen decomposition $A = V \Lambda V^{-1}$ → invertible

Λ → diagonal

x-decomposition $A = U \Sigma V^{-1} = U \Sigma V^{T}$ → orthogonal

U → orthogonal

Σ → diagonal

A new decomposition!

$y = Ax \implies$ Change of basis $\vec{y} = x_1' \vec{u_1} + y_2' \vec{u_2} + \cdots + y_m' \vec{u_n}$

$$\vec{y} = U y' \implies y' = U^T y$$

$$x = V x' \implies x' = V^T x$$

$$y' = \Sigma x'$$

diagonal

$$U^T y = \Sigma V^T x \xrightarrow[y = Ax]{} U^T A x = \Sigma V^T x \quad \text{for all } x \in \mathbb{R}^n.$$

$$\implies U^T A = \Sigma V^T$$

$$\implies AV = U\Sigma$$

$$y = Ax \implies U y' = A V x' \xrightarrow[y' = \Sigma x']{} U \Sigma x' = A V x' \quad \text{for all } x'$$

$$\implies U\Sigma = AV$$

$$AV = U\Sigma \implies \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_l \end{bmatrix} \quad l = \min(m,n)$$

A new decomposition!

K. N. Toosi
University of Technology

$$AV = U\Sigma \Rightarrow \begin{bmatrix} A \end{bmatrix}\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_l \end{bmatrix} \quad l = \min(m,n)$$

$$i \le \min(m,n) \Rightarrow A v_i = \sigma_i u_i$$

u_i → right singular vector

σ_i → singular value

Av_i → left singular vector

A new decomposition!

Eigenvalue $\quad Av_i = \lambda_i v_i \quad$ there are ℓ $(1 \leq \ell \leq n)$ eigenvalu

Singular value $\quad Av_i = \sigma_i u_i \quad$ $\left.\begin{array}{l} u_1, \ldots, u_m \\ v_1, \ldots, v_m \end{array}\right\}$ ~~orthogonal~~ orthonormal

Example

K. N. Toosi
University of Technology

$$AV = U\Sigma \qquad A \in \mathbb{R}^{3\times 2}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

$$Av_1 = \sigma_1 u_1$$

$$Av_2 = \sigma_2 u_2$$

Eigen Decomposition vs Singular Value Decomposition

Eigenvalue $AV = V\Lambda \xRightarrow{\text{diagonalizable}} A = V\Lambda V^{-1}$

Singular Value $AV = U\Sigma \xRightarrow[\text{always}]{} A = U\Sigma V^T$

Singular Value Decomposition (SVD)

K. N. Toosi
University of Technology

$$A = U \Sigma V^T$$: Singular Value Decomposition (SVD)

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)}$$

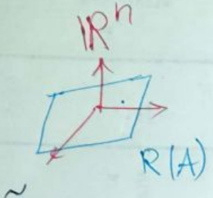
U, V orthogonal

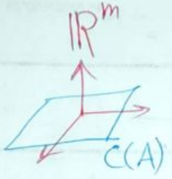
SVD and the relation between the row space and col

LA 27

$A \in \mathbb{R}^{m\times n}$ $\quad$ $\operatorname{rank}(A)=r$

$f(x)=Ax$ $\quad$ $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$



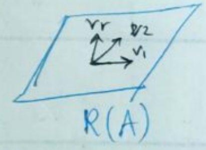


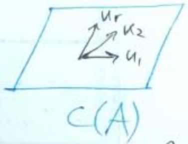
$\tilde{f}(x)=Ax$ $\quad$ $\tilde{f}: R(A) \longrightarrow C(A)$

$\forall x \in R(A) \quad \tilde{f}(x)=f(x)$

$\dim(R(A)) = \dim(C(A)) = r$

$\tilde{f}$ is one-to-one & onto (invertible)





$v_1, v_2, \dots, v_r$ orthonormal basis for $R(A)$

$u_1, u_2, \dots, u_r$ orthonormal " for $C(A)$