Linear Algebra for Computer Science

Lecture 12

General LU decomposition, Null Space

basic LU decomposition my not exist!

LU elim

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$\boxed{E} A_{n\times n} = L U =$

$E_2 E_2 E_1 A_{n\times n} = U$

$a g$

$$\begin{array}{ccc} & L & U \\ \begin{bmatrix}0&3&7\\1&4&8\\2&5&16\end{bmatrix} = & \begin{bmatrix}a&0&0\\b&c&0\\d&e&f\end{bmatrix} & \begin{bmatrix}g&h&i\\0&j&k\\0&0&l\end{bmatrix}\end{array}$$

Row exchange - permutation matrix

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$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 7 \\ 1 & 4 & 8 \\ 2 & 5 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 8 \\ 0 & 3 & 7 \\ 2 & 5 & 6 \end{bmatrix}$$

Permutation matrices

$a \quad b \quad c$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

3×3

a b c
b a c
b c a
a c b
c a b
c b a

Permutation matrices

there are $n!$ permutation matrices of dimension $n \times n$.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$r3 \to 1$
$r1 \to 2$
$r2 \to 3$

$$P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = P^T$$

for permutation matrices we have $P^{-1} = P^T$

$$P^T P = P P^T = I$$

P is an orthogonal matrix

LU Decomposition - general form

General LU decomposition of $A \in \mathbb{R}^{n \times n}$

$$PA = LU$$

permutation matrix

LDU decomposition

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LDU decomposition

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 24 & 2 & 0 \\ 12 & 8 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & \frac{1}{2} & 7 \\ 0 & 0 & 5 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 14 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & \\ 8 & 1 & \\ 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} 6 & & \\ & 1 & \\ & & 20 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ & 1 & 14 \\ & & 1 \end{bmatrix}$$

$$A = \quad L \quad D \quad U$$

pivot and floating point arithmetic

$$\begin{bmatrix} 10^{-7} & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

floating Point

Back to linear equations

$Ax = b$

A is square & non-singular

What about the singular or non-square case?

General Linear Equations

A not of full-rank

A not square

Does $A x = b$ have a solution?

Null space

A column space $C(A)$ (range A)

row space $R(A)$

null space $N(A)$ (kernel A)

$f(x)$ $\quad$ $\text{range}(f) = \{ f(x) \mid x \in \mathcal{X} \}$ $\quad$ $f(x) = Ax$

$f: \mathcal{X} \to \mathcal{Y}$

$\text{kernel}(f) = \{ x \in \mathcal{X} \mid f(x) = 0 \}$

$N(A) = \{ x \mid Ax = 0 \}$ $\quad$ فضای تهی

null space $\quad$ $A \in \mathbb{R}^{m \times n}$

Null space is a linear subspace

$$N(A) = \{ x \mid Ax = 0 \}$$

فضای تهی

نول

null space

$$A \in \mathbb{R}^{m \times n}$$

$$y, x \in N(A) \Rightarrow \alpha x \overset{?}{\in} N(A)$$

$$x + y \overset{?}{\in} N(A)$$

$$A(\alpha x) = \alpha A x = \alpha \cdot 0 = 0 \Rightarrow \alpha x \in N(A)$$

$$A(x+y) = Ax + Ay = 0 + 0 = 0 \Rightarrow x + y \in N(A)$$

$\Rightarrow N(A)$ is a linear subspace (of $\mathbb{R}^n$)

$Ax = 0$

$Ax = 0$ $\quad$ $0 \in N(A)$ $\quad$ $\boxed{\alpha H}$

$N(A)$ includes all answers to $\boxed{Ax=0}$

Computing the null space

$$\underbrace{\begin{bmatrix} 1 & 5 & 6 \\ 2 & 6 & 8 \\ 3 & 7 & 10 \\ 4 & 8 & 12 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in N(A) \qquad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in N(A)$$

$$N(A) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} x + \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} y + \begin{bmatrix} 6 \\ 8 \\ 10 \\ 12 \end{bmatrix} z = 0 \right\}$$

$$\begin{bmatrix} \alpha \\ \alpha \\ -\alpha \end{bmatrix} \in N(A) \qquad \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \in N(A)$$

Computing the null space

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$$\underbrace{\begin{bmatrix}1&5&6\\2&6&8\\3&7&10\\4&8&12\end{bmatrix}}_{A}\begin{bmatrix}x\\y\\z\end{bmatrix}=0$$

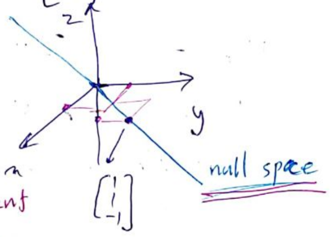
$$\begin{bmatrix}0\\0\\0\end{bmatrix}\in N(A)$$

$$\begin{bmatrix}1\\1\\-1\end{bmatrix}\in N(A)$$

$$N(A)=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\ \middle|\ \begin{bmatrix}1\\2\\3\\4\end{bmatrix}x+\begin{bmatrix}5\\6\\7\\8\end{bmatrix}y+\begin{bmatrix}6\\8\\10\\12\end{bmatrix}z=0\right\}$$

$$\begin{bmatrix}\alpha\\\alpha\\-\alpha\end{bmatrix}\in N(A) \qquad \alpha\begin{bmatrix}1\\1\\-1\end{bmatrix}\in N(A)$$

$$\dim(N(A))=\underbrace{3}_{\text{No. of columns}}-\underbrace{2}_{\substack{\text{No. independent columns}\\=\text{rank}(A)}}=1$$



Elimination does not change the null space

for any **invertible** E we have $A\,x = 0 \Leftrightarrow E\,A\,x = 0$

Find null space by elimination - row echelon form

$$\begin{bmatrix} 1 & 5 & 6 \\ 2 & 6 & 8 \\ 3 & 7 & 10 \\ 4 & 8 & 12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 6 \\ 0 & -4 & -4 \\ 0 & -8 & -8 \\ 0 & -12 & -12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 6 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

row echelon form

$$Ax = b \quad \Updownarrow \quad EAx = Eb$$

F invertible

$$Ax = 0 \quad \Updownarrow \quad EAx = 0$$

$$\begin{bmatrix} 1 & 5 & 6 \\ 0 & -4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Find null space by elimination - row echelon form

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Pivots

$$\begin{bmatrix} \boxed{1} & 5 & 6 \\ 0 & \boxed{-4} & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 5 \\ -4 \end{bmatrix} y + \begin{bmatrix} 6 \\ -4 \end{bmatrix} z = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x + 5y \\ -4y \end{bmatrix} = \begin{bmatrix} -6z \\ 4z \end{bmatrix}$$

free variable

$$-4y = 4z \Rightarrow y = -z$$

$$x + 5y = -6z \Rightarrow x - 5z = -6z \Rightarrow x = -z$$

Null space: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = N(A)$