Linear Algebra for Computer Science

Lecture 15

Overdetermined Systems, Least Squares

Underdetermined system

1

$$\underline{\underline{Ax=b}}$$

$$\begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} x \end{bmatrix}=\begin{bmatrix} b \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} x \end{bmatrix}=\begin{bmatrix} b \end{bmatrix}$$

$A \in \mathbb{R}^{m\times n}$

$r = \text{Rank}(A) < n$
$b \in C(A)$ $\Rightarrow$

$r < n$ independen equations
n unknown

<u>Underdetermined</u> Equations

Overdetermined system

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$$Ax = b$$

$$\begin{bmatrix} \; \\ \; \\ \; \\ \; \\ \; \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

$$A \in \mathbb{R}^{m \times n}$$

$$b \notin C(A)$$

Overdetermined

Underdetermined/Overdetermined

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In some texts, under-/over-determined simply means less/more equations than unknowns.

Linear measurements

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$$y_1 = a_{11} x_1 + a_{12} x_2 + a_{13} x_3$$
$$y_2 = a_{21} x_1 + a_{22} x_2 + a_{23} x_3$$
$$y_3 = a_{31} x_1 + a_{32} x_2 + a_{33} x_3$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

measurement

Noisy linear measurements

$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + n_1$ → (measurement process) noise

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + n_2$

$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + n_3$

$y_4 = a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + n_4$

$y_5 =$

$y_6 =$

$y = Ax + n$ ← noise

$n \in \mathbb{R}^m$

II

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} n \end{bmatrix}$$

noise is unknown

noise is (relatively) small

$y \notin C(A)$

$y - n \in C(A)$

$\boxed{Ax = y} \rightarrow$ overdetermind

$A \in \mathbb{R}^{m \times n}$

What is the best solution?

$y \notin C(A)$

$y - n \in C(A)$

$\boxed{Ax = y} \to$ overdetermind

$A \in \mathbb{R}^{m \times n}$

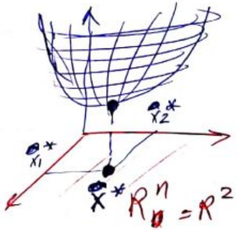
there is no x such that $A\underline{x} = y$

find an x such that $\|Ax - y\|$ is small

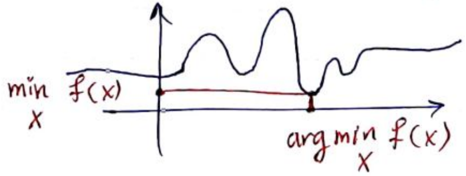
$\|u\| =$ lenght of u.

$Ax - y \in \mathbb{R}^m$

$f(x) = \|Ax - y\|$ $\quad f: \mathbb{R}^n \to \mathbb{R}$



$x^* = \underset{x}{\arg\min}\ f(x)$

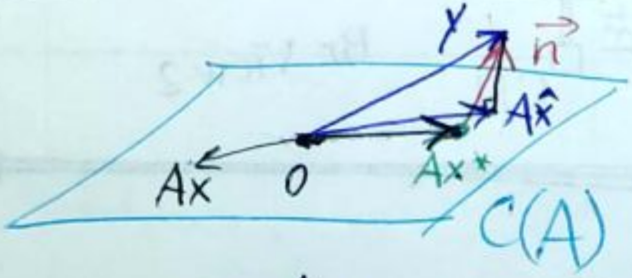


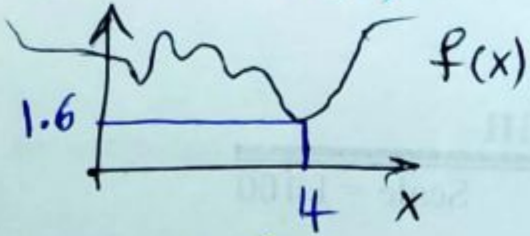
What is the best solution?

$$y = Ax^* + \vec{n}$$

$$y \simeq Ax^*$$

$$\hat{x} = \underset{x}{\operatorname{argmin}} \| Ax - y \|$$





$$1.6 = \min_{x} f(x)$$

$$4 = \underset{x}{\operatorname{argmin}} f(x)$$

What is the best solution?

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$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} n \end{bmatrix}$$

$Ax = y$ overdetermined

III

$$x^* = \underset{x}{\operatorname{argmin}} \; \| Ax - y \|$$

Length of a vector

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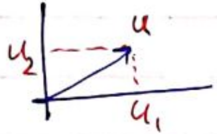
$$\begin{bmatrix}u_1\\u_2\end{bmatrix}=u$$



$$\|u\| = \sqrt{u_1^2+u_2^2}$$

$$u=\begin{bmatrix}u_1\\u_2\\\vdots\\u_m\end{bmatrix} \Rightarrow \|u\| = \|u\|_2 = \sqrt{u_1^2+u_2^2+\cdots+u_m^2}$$

$$= \left(\sum_{i=1}^{m} u_i^2\right)^{1/2}$$

$$= \left(\langle u,u\rangle\right)^{1/2}$$

$$= (u^T u)^{1/2}$$

$$= \sqrt{u^T u}$$

$$\|u\|^2 = \underline{u^T u}$$

Length of a vector - complex case

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{C}^m \qquad u_i \text{: complex}$$

$$\|u\| = \sqrt{\sum |u_i|^2} = \langle u, u \rangle$$

$$|u_i| = u_i \, \overline{u_i}$$

$$\langle u, v \rangle = \sum u_i \overline{v_i} = V^H u$$

Best solution?

$$\|u\|^2 = u^T u$$

$$x^* = \arg\min_x \|Ax - b\|$$

$$= \arg\min_x \|Ax - b\|^2 = (Ax-b)^T(Ax-b)$$

$$= (x^T A^T - b^T)(Ax - b) = x^T A^T A x - b^T A x - x^T A^T b + b^T b$$

GR

$$\underbrace{\underset{1\times n}{x^T}\ \underset{n\times m}{A^T}\ \underset{m\times n}{A}\ \underset{n\times 1}{x}}_{1\times 1} \qquad \underbrace{\underset{1\times m}{b^T}\ \underset{m\times n}{A}\ \underset{n\times 1}{x}}_{1\times 1}$$

XOR

$$= x^T(A^T A)x - 2(b^T A)x + b^T b$$

Least Squares Solution

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IV 16

$Ax = y$ A has full column rank

$$\begin{bmatrix} A \end{bmatrix} x = \begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$x^* = \operatorname{argmin} \|Ax - y\|^2 = \underset{x}{\operatorname{argmin}} \left\| \begin{bmatrix} a_1^T x - y_1 \\ a_2^T x - y_2 \\ \vdots \\ a_m^T x - y_m \end{bmatrix} \right\|^2$$

$$= \underset{x}{\operatorname{argmin}} \underbrace{\underbrace{(a_1^T x - y_1)^2 + (a_2^T x - y_2)^2 + \cdots + (a_m^T x - y_m)^2}_{\text{sum of squares} \quad \text{مجموع مربعات}}}_{\text{Least Squares} \quad \text{مسأله کمینه (مجموع) مربعات}}$$

x^*: Least Squares Solution

b $e = b - Ax^*$

Geometric Interpretation

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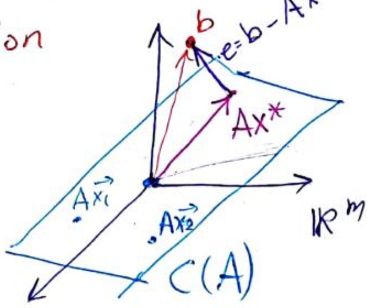
Least Squares   مسأله کمینه (مجموع) مربعات

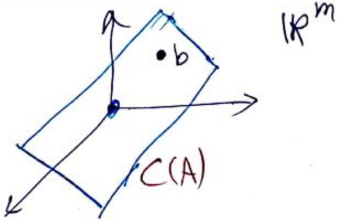
x^*: Least Squares Solution

$$Ax = b \qquad b \notin C(A)$$

$$\min_x \|Ax-b\| = \text{dist}(b, C(A))$$



$$\boxed{b \in C(A) \Rightarrow \min_x \|Ax-b\| = 0}$$



$$\Rightarrow (b-Ax^*) \perp Ax^*$$

$$(b-Ax^*) \perp Ax$$

for all x

Orthogonal vectors

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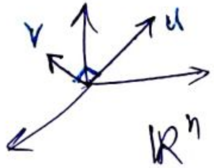
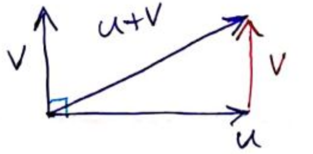




قضیهٔ فیثاغورث : $\quad u \perp v \Rightarrow \|u\|^2 + \|v\|^2 = \|u+v\|^2$

$$u^Tu + v^Tv = (u+v)^T(u+v)$$

$$= (u^T+v^T)(u+v)$$

$$= u^Tu + \underbrace{u^Tv + v^Tu}_{2u^Tv} + v^Tv \in \mathbb{R}$$

$$\Rightarrow \quad u \perp v \Leftrightarrow u^T\cancel{u} + \cancel{v^T}v = u^T\cancel{u} + 2u^Tv + \cancel{v^T}v$$

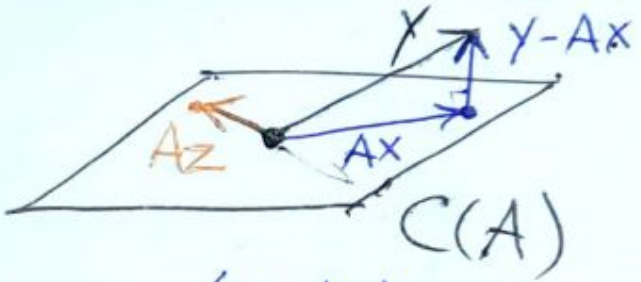
$$\Rightarrow 2u^Tv = 0 \Rightarrow u^Tv = 0 = \langle u, v \rangle$$

$$\boxed{u \perp v \Leftrightarrow \langle u, v \rangle = 0}$$

Least Squares

$$\underbrace{y = Ax}_{m\times n} \qquad A\in\mathbb{R}^{m\times n}$$

$$m > n$$



$$(y-Ax)\perp Ax$$

$$(y-Ax)\perp Az \quad \forall z\in\mathbb{R}^n$$

$$\forall z\in\mathbb{R}^n \quad (y-Ax)\perp Az \Rightarrow \underbrace{\underbrace{(y-Ax)^T}_{1\times m}\underbrace{(Az)}_{m\times 1}}_{1\times 1} = 0$$

$$\left((y-Ax)^T A\right) z = 0 \quad \text{for all } z\in\mathbb{R}^n$$

Least Squares

$$\underset{m\times n}{M} z = 0 \quad \forall z \qquad M \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = 0 \qquad M \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = \begin{bmatrix}0&0&0&0\\ \\ \\ \end{bmatrix}$$

$$e_1$$

$$MI = 0_{m\times n} \Rightarrow M = 0$$

Least Squares

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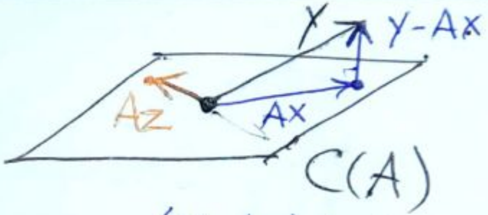
$\underbrace{y = Ax}_{m\times n}$ $\quad A\in\mathbb{R}^{m\times n}$

$m > n$



$(y-Ax)\perp Ax$

$(y-Ax)\perp Az \quad \forall z\in\mathbb{R}^n$

$\forall z\in\mathbb{R}^n \quad (y-Ax)\perp Az \Rightarrow \underbrace{\underbrace{(y-Ax)^T}_{1\times m}\underbrace{(Az)}_{m\times 1}}_{1\times 1} = 0$

$\left((y-Ax)^T A\right) z = 0$ for all $z\in\mathbb{R}^n$

$\left((y-Ax)^T A\right) z = 0 \Rightarrow (y-Ax)^T A = 0_{1\times n} \in \mathbb{R}^{1\times n}$

$\Rightarrow A^T(y-Ax) = 0_{n\times 1} \Rightarrow A^T y = \underbrace{\underbrace{A^T}_{n\times m}\underbrace{A}_{m\times n}}_{n\times n}x \Rightarrow x = (A^TA)^{-1}A^Ty$

least squares solution

جواب کمینه مربعات