$Ax=b$ $\quad A\in\mathbb{R}^{m\times n}$
$x\in\mathbb{R}^{n}$
$b\in\mathbb{R}^{m}$

$b\notin C(A)$ no solution

A has full column rank $\Rightarrow n$ independent columns
$\Rightarrow \text{rank}(A)=n \Rightarrow n$ " row $\Rightarrow m\geq n$

$\Rightarrow$ choose $n$ independents rows of $A$.

$\Rightarrow$ Assume, without losing generality, that the first $n$ rows of $A$ are independent

$$\underset{m\times n}{A}=\begin{bmatrix} Q \\ P \end{bmatrix}$$

$Q$ $n\times n \Rightarrow Q$ non-singular $\Rightarrow$ invertible
$P$ $(m-n)\times n$

$$Ax=b$$

$$b=\begin{bmatrix} u \\ v \end{bmatrix}\begin{matrix}\}n \\ \}m-n\end{matrix} \qquad \begin{matrix} n\{ \\ m-n\{ \end{matrix}\begin{bmatrix} Q \\ P \end{bmatrix}x=\begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{cases} Qx=u & \text{(I)} \\ \underline{Px=v} & \text{(II)} \end{cases}$$

invertible

$x^*=Q^{-1}u$ (if $Ax=b$ has a solution)

$Px^*=v \Rightarrow x^*$ is a solution to $Ax=b$ $\quad \begin{bmatrix} Q \\ P \end{bmatrix}x=\begin{bmatrix} u \\ v \end{bmatrix}$

$Px^*\neq v \Rightarrow Ax=b$ has no solution
$b\notin C(A)$

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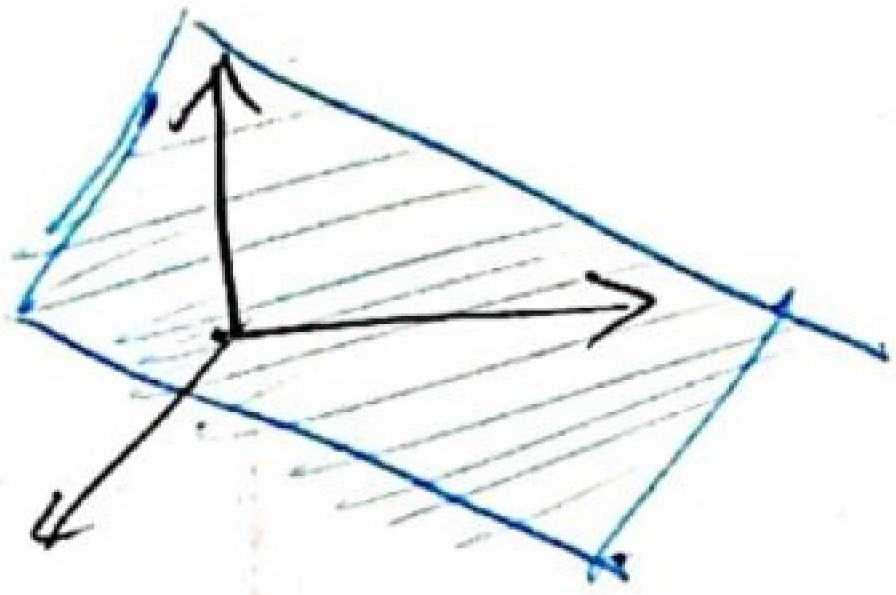
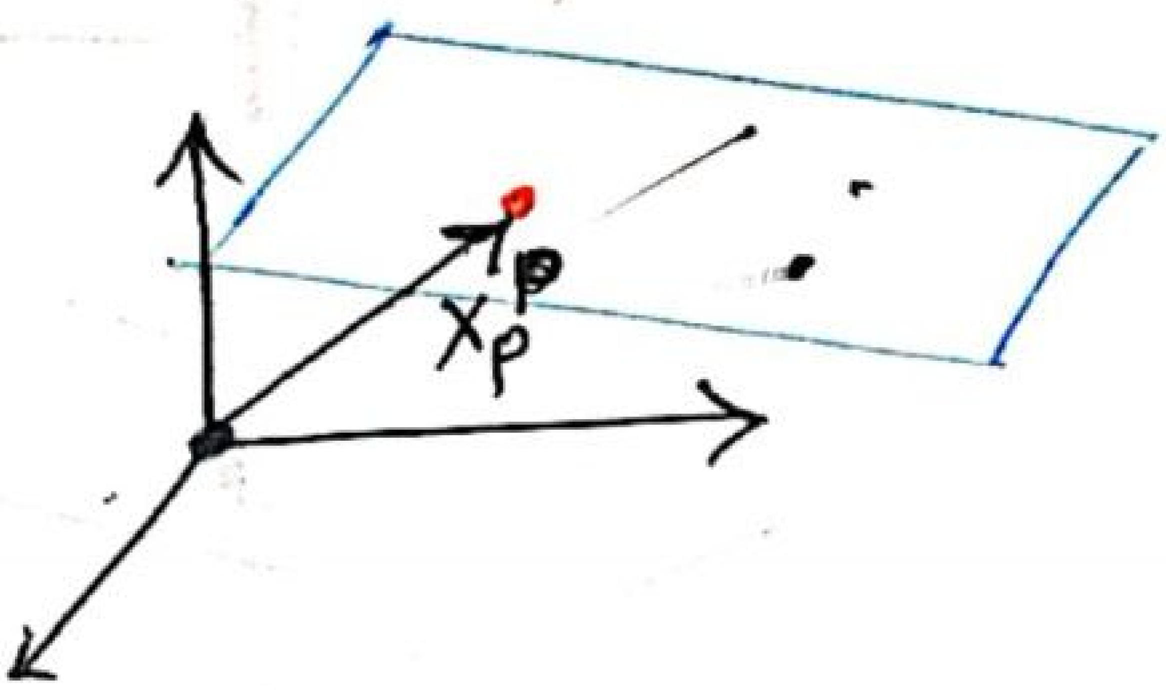
$Ax=b$ $\quad A\in\mathbb{R}^{m\times n}$ and $x, b$ are vetors of compatible size.

$S=\{x \mid Ax=b\}=$ all solutions to $Ax=b$

Is $\underline{S}$ a linear subspace? $\Rightarrow u,v\in S \overset{?}{\Rightarrow} \alpha u+\beta v \overset{?}{\in} S$

$u,v\in S \Rightarrow \begin{cases} Au=b \\ Av=b \end{cases}$ $\quad A(\alpha u+\beta v)=\alpha Au+\beta Av$
$=\alpha b+\beta b=(\alpha+\beta)b$






Affine Subspace

$S$ is an affine subspace if $\exists\, u \in S$ such that $S - u = \{x - u \mid x \in S\}$ is a linear subspace.

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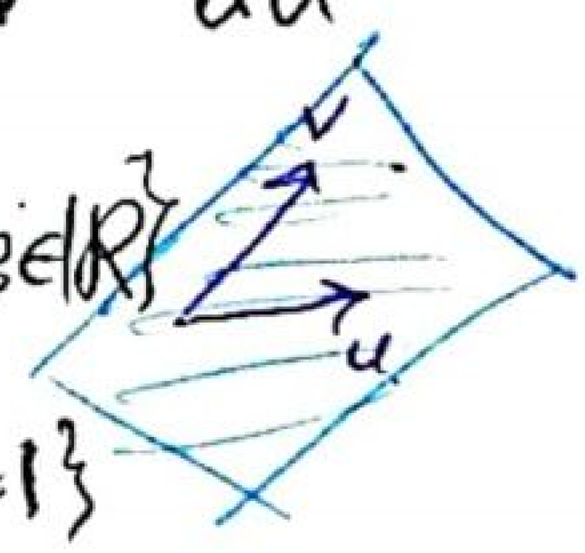
$Au = b$
$Av = b$ $\Rightarrow$ ~~Au~~ $A(\alpha u + \beta v) = (\alpha + \beta) b \neq b$

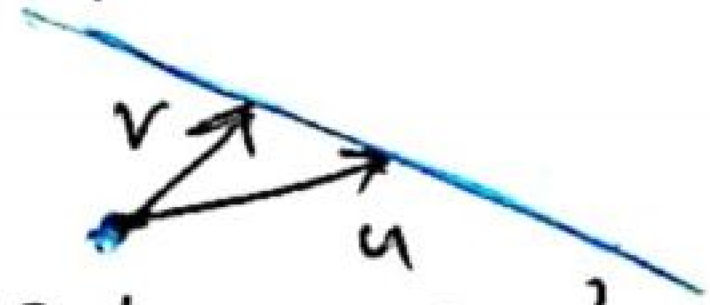
$\alpha + \beta = 1$

~~p~~ $u, v \in V$ linear combination of $u, v$ $\alpha u$

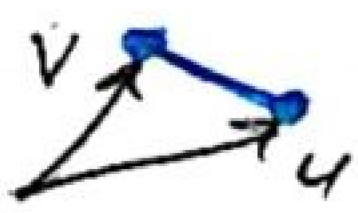



linear combination of $u, v = \{\alpha u + \beta v \mid \alpha, \beta \in \mathbb{R}\}$

affine combination of $u, v = \{\alpha u + \beta v \mid \alpha + \beta = 1\}$



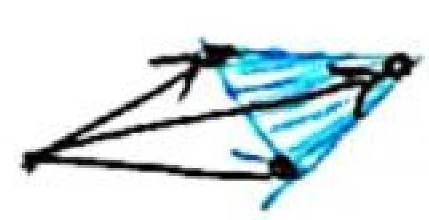

convex combination of $u, v = \{\alpha u + \beta v \mid \alpha + \beta = 1,\ \alpha, \beta \geq 0\}$




$u_1, \dots, u_n$ lin comb $\{\sum \alpha_i u_i \mid \alpha_i \in \mathbb{R}\}$

aff comb $\{\alpha \sum \alpha_i u_i \mid \sum_{i=1}^{n} \alpha_i = 1\}$

conv comb $\{\sum \alpha_i u_i \mid \sum \alpha_i = 1, \alpha_i \geq 0\}$



$Ax = b \quad A \in \mathbb{R}^{m\times n}$

A has full column rank $\quad m \geq n \quad \begin{bmatrix} A \end{bmatrix}$

$\Rightarrow rank(A) = n$

$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \Rightarrow a_1, a_2, \dots, a_n$ are linearly independent

$b \notin C(A) \Rightarrow$ no solutions.

$b \in C(A) \Rightarrow$ solutions $\{x_p + x_n \mid x_n \in N(A)\}$

$Ax = 0 \Rightarrow \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$

$\Rightarrow x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = 0 \Rightarrow x_1 = x_2 = \cdots = x_n = 0$

$x = \begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix} = \vec{0} \Rightarrow N(A) = \{\vec{0}\}$ $\quad \boxed{x_p' = x_p + \underset{0}{x_n}}$

$\Rightarrow Ax = b$ has a unique solution

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$Ax = b \quad A$: has full row rank $\in \mathbb{R}^{m\times n}$

$m \leq n$

$\begin{bmatrix} A \end{bmatrix} \quad m \leq n \quad rank(A) = m$

$\left.\begin{matrix} C(A) \subseteq \mathbb{R}^m \\ \dim(C(A)) = rank(A) = m \end{matrix}\right\} C(A) = \mathbb{R}^m$

$\Rightarrow b \in C(A)$

at least on solution

one or infinite no. of solutions

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$A \in \mathbb{R}^{m\times n}$

A full coulmn rank $\Rightarrow m \geq n$ no or one solution.

A full row rank $\Rightarrow m \leq n$ infinite or one solution

both $m = n$, A non-singular $\Rightarrow$ one solution

LA14 (IV)

$$A=\begin{bmatrix}1&2&3&4&5\\2&4&6&8&10\\3&6&10&2&4\\4&8&13&6&9\end{bmatrix}\qquad x^*=\begin{bmatrix}1\\-1\\2\\3\\1\end{bmatrix}$$

$$Ax=\begin{bmatrix}1&2&3&4&5\\2&4&6&8&10\\3&6&10&2&4\\4&8&13&6&9\end{bmatrix}x=\underbrace{\begin{bmatrix}12\\24\\19\\31\end{bmatrix}}_{b}\qquad b'=\begin{bmatrix}12\\24\\19\\32\end{bmatrix}$$

b → جواب دارد

b' → جواب ندارد

$$\left[\begin{array}{ccccc|cc}1&2&3&4&5&12&12\\2&4&6&8&10&24&24\\3&6&10&2&4&19&19\\4&8&13&6&9&31&32\end{array}\right]$$

$$\begin{array}{l}r_2-=2r_1\\r_3-=3r_1\end{array}\left[\begin{array}{ccccc|cc}1&2&3&4&5&12&12\\0&0&0&0&0&0&0\\0&0&1&-10&-11&-17&-17\end{array}\right]$$

$$\Downarrow GJ$$

$$\begin{bmatrix}\boxed{1}&2&0&34&38\\0&0&\boxed{1}&-10&-11\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\\t\\u\end{bmatrix}=\begin{bmatrix}97\\-17\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x\\z\end{bmatrix}+\begin{bmatrix}2&34&38\\0&-10&-11\end{bmatrix}\begin{bmatrix}y\\t\\u\end{bmatrix}=\begin{bmatrix}97\\-17\end{bmatrix}$$

$$y=t=u=0\Rightarrow\begin{bmatrix}x\\z\end{bmatrix}=\begin{bmatrix}97\\-17\end{bmatrix}\Rightarrow x_p=\begin{bmatrix}97\\0\\-17\\0\\0\end{bmatrix}$$

null space
$$N=\begin{bmatrix}-2&34&38\\1&0&0\\0&10&11\\0&1&0\\0&0&1\end{bmatrix}$$