$Ax = b$ $\quad A \in \mathbb{R}^{m \times n}$

$x \in \mathbb{R}^n$

m equations $\quad b \in \mathbb{R}^m$

n unknown

~~under d~~

$m < n$ $\quad [\ A\ ][x] = [b]$ $\quad \infty$ solutions

x cannot be determined,

$\Rightarrow$ underdetermined system

$m > n$ $\quad [\ A\ ][x] = [b]$ $\quad$ overdetermined

$b \notin C(A)$

$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + n_1$

$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + n_2$

$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + n_3$

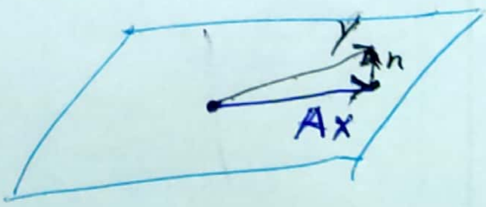
$y_4 = a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + n_5$

$y_5 = a_{51}x_1 + a_{52}x_2 + a_{53}x_3 + n_6$

$y = A\vec{x} + \vec{n}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ a & a & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a & a & a \\ a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix}$$



$y = \underline{A}\, x + \vec{n}$ $\qquad y \simeq Ax$

$Ax \quad y$

$\vec{x}^*$: actual $\vec{x}$

$A \in \mathbb{R}^{m \times n}$

$\vec{y} = A\vec{x}^* + \vec{n}$ $\qquad \vec{y} = \checkmark$

$y \in \mathbb{R}^m \quad Ax \in \mathbb{R}^m$ $\qquad A = \checkmark$

$\vec{n}$ small

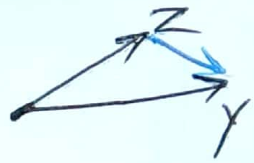
$y \simeq Ax^*$ $\qquad \vec{n} = ?$

Solution 1: find a bunch of candidate solutions $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p$

choose the one with minimum ~~dist~~

distace(Ax_i, y)

Euclean Distance



$$D(y, z) = D\left(\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}\right)$$

$$= \sqrt{(y_1 - z_1)^2 + (y_2 - z_2)^2 + \cdots + (y_m - z_m)^2}$$

$\|v\| = \sqrt{v^T v}$
$= \sqrt{v_1^2 + \cdots + v_n^2}$

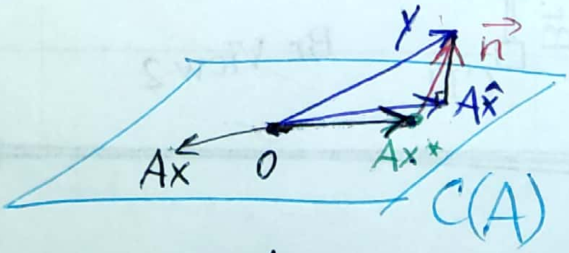
$$= \|y - z\| = \sqrt{(y-z)\cdot(y-z)}$$

$$\sqrt{(y-z)^T (y-z)}$$

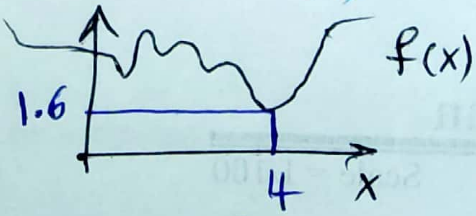
np.linalg.norm(y-z)

$$y = Ax^* + \vec{n}$$

$$y \simeq Ax^*$$

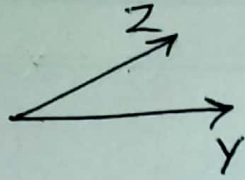


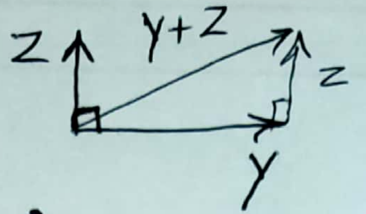
$$\hat{x} = \underset{x}{\operatorname{argmin}} \|Ax - y\|$$



$$1.6 = \min_x f(x)$$

$$4 = \underset{x}{\operatorname{argmin}}\, f(x)$$





$$y \perp z \Rightarrow \|y\|^2 + \|z\|^2 = \|y+z\|^2$$

$$y^T y + z^T z = (y+z)^T (y+z)$$

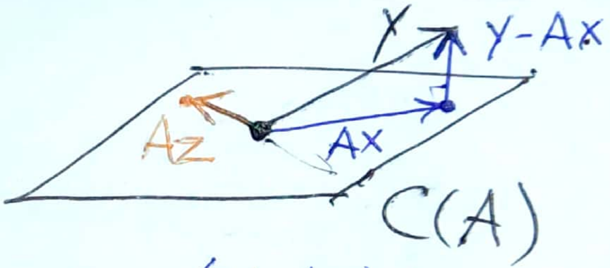
$$\cancel{y^T y} + \cancel{z^T z} = \cancel{y^T y} + y^T z + \underbrace{z^T y}_{y^T z} + \cancel{z^T z}$$

$$2\, y^T z = 0 \Rightarrow y^T z = 0$$

$$y \perp z = \langle y, z \rangle = y_1 z_1 + y_2 z_2 + \cdots + y_m z_m$$
$$= y^T z = 0$$

$$\underset{m \times n}{y = Ax} \quad A \in \mathbb{R}^{m \times n}$$

$$m > n$$



$$(y - Ax) \perp Ax$$

$$(y - Ax) \perp Az \quad \forall z \in \mathbb{R}^n$$

$$\forall z \in \mathbb{R}^n \quad (y - Ax) \perp Az \Rightarrow \underbrace{\underbrace{(y-Ax)^T}_{1 \times m} \underbrace{(Az)}_{m \times 1}}_{1 \times 1} = 0$$

$$\left((y - Ax)^T A\right) z = 0 \quad \text{for all } z \in \mathbb{R}^n$$

$$\boxed{\underset{m \times n}{M} z = 0 \quad \forall z \qquad M \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \qquad M \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \qquad MI = 0_{m \times n} \Rightarrow M = 0}$$

e_1

$$\left((y - Ax)^T A\right) z = 0 \Rightarrow (y - Ax)^T A = 0_{1 \times n} \in \mathbb{R}^{1 \times n}$$

$$\Rightarrow A^T (y - Ax) = 0_{n \times 1} \Rightarrow A^T y = \underbrace{\underset{n \times m}{A^T} \underset{m \times n}{A}}_{n \times n} x \Rightarrow x = (A^T A)^{-1} A^T y$$

least squares solution

جواب کمینه مربعات