LA16 (I)

$$Ax = b \qquad \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad a_i^T x = b_i$$

معادلات

$A \in \mathbb{R}^{m \times n}$

$x \in \mathbb{R}^n$

$b \in \mathbb{R}^m$

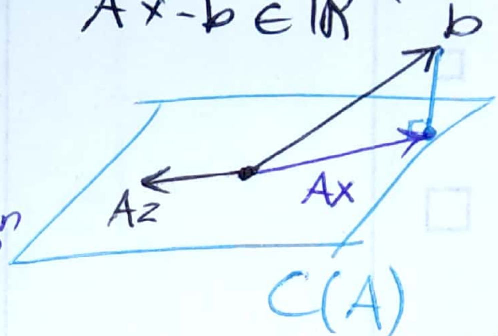
over determined $m > n$, $b \notin C(A)$

$\nexists\ Ax = b \qquad Ax \simeq b \qquad Ax \in \mathbb{R}^m \quad b \in \mathbb{R}^m$

$Ax - b \in \mathbb{R}^m$

$$x_{opt} = \underset{x}{\operatorname{argmin}} \|\boxed{Ax} - b\|$$

→ least squares problem



$(b - Ax) \perp Az \quad \forall z \in \mathbb{R}^n$

$$(Az)^T (b - Ax) = 0 \Rightarrow z^T A^T (b - Ax) = 0 \quad \forall z$$

$$\Rightarrow A^T(b - Ax) = 0 \Rightarrow A^T b - A^T A x = 0$$

$$\Rightarrow (\underbrace{\underbrace{A^T}_{n \times m} \underbrace{A}_{m \times n}}_{n \times n}) x = \underbrace{\underbrace{A^T}_{n \times m} \underbrace{b}_{m \times 1}}_{n \times 1}$$

$\mathbb{R}^n$

$\cancel{A^{-1}}$ $\text{rank}(A) = n$

$\Rightarrow A^T A$ non-singular

n^{α}

$$x_{opt} = (A^T A)^{-1} A^T b \qquad \underline{Ax \simeq b}$$

least square solution

$\cancel{Ax \simeq b}$ $\quad x_{opt} = \underset{x}{\operatorname{argmin}} \|Ax - b\| = \underset{x}{\operatorname{argmin}} \|Ax - b\|^2$

least squares problem

$$\underset{m \times n}{\|Ax - b\|^2} = \left\| \begin{bmatrix} -a_1^T- \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} x - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} a_1^T x - b_1 \\ a_2^T x - b_2 \\ \vdots \\ a_m^T x - b_m \end{bmatrix} \right\|^2$$

$$= \underbrace{(a_1^T x - b_1)^2 + (a_2^T x - b_2)^2 + \cdots + (a_m^T x - b_m)^2}_{\text{sum of squared terms}}$$

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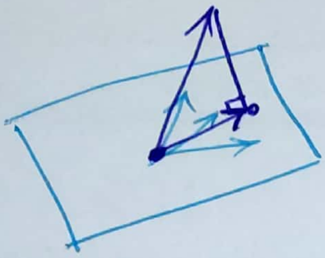
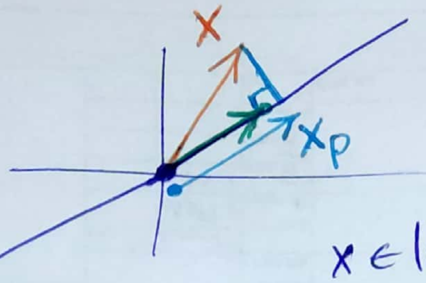
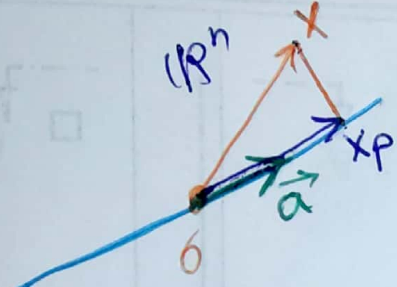
LA16 II





$x \in \mathbb{R}^n$



$x - x_p \perp a \implies (x - x_p)^T a = 0$

$x_p = \alpha \vec{a} = \boxed{\vec{a}\,\alpha}$

$(x - x_p)^T a = 0 \implies \underbrace{x^T\vec{a}} = x_p^T \vec{a} = \underbrace{(\alpha\vec{a})^T\vec{a}}$

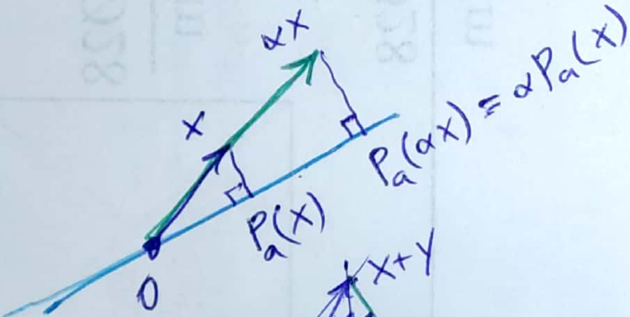
$\implies x^T\vec{a} = \alpha\, \vec{a}^T\vec{a} \implies \alpha = \dfrac{x^T a}{a^T a} = \dfrac{\langle x, a\rangle}{\langle a, a\rangle} = \dfrac{\langle x, a\rangle}{\|a\|^2}$

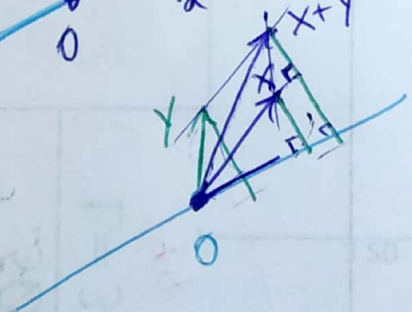
$x_p = \alpha a = \left(\dfrac{x^T a}{a^T a}\right)\vec{a}$

$x_p = P_a(x) = \left(\dfrac{x^T a}{a^T a}\right)\vec{a}$

P_a: projection onto span($\vec{a}$) ; $\dfrac{x^T a}{a^T a} \in \mathbb{R}$

$P_a : \mathbb{R}^n \to \mathbb{R}^n$





$P_a(x+y) = P_a(x) + P_a(y)$

P_a is a linear map

$\implies \boxed{P_a\ \mathbb{R}^n \to \mathbb{R}^n}$ $P_a(x) = x$

$\implies \exists Q \in \mathbb{R}^{n\times n}$ $P_a(x) = Qx$

$P_a(x) = \dfrac{x^T a}{a^T a}\, a$

$\alpha \in \mathbb{R}$
$v \in \mathbb{R}^n$

$\alpha V \rightarrow$ not a
α: 1×1, V: $n\times1$

$\alpha V \rightarrow$ not a matrix product
α: 1×1, V: $n\times1$

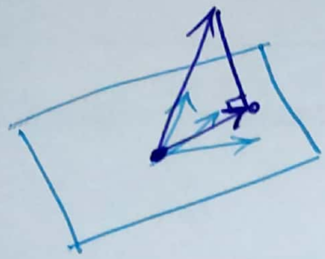
$V\alpha \rightarrow$ product of an $n\times1$ by a 1×1 matrix
V: $n\times1$, α: 1×1

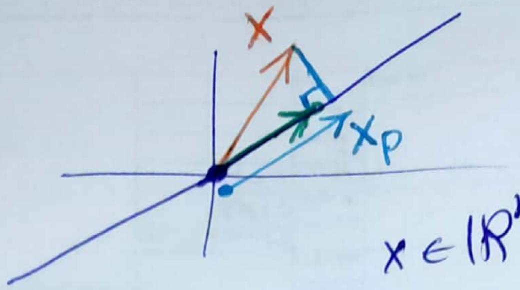
$\alpha = x^T y$, $x, y \in \mathbb{R}^m$

$\alpha V = (x^T y) V \neq x^T \underset{m\times1}{y}\, \underset{n\times1}{V}$

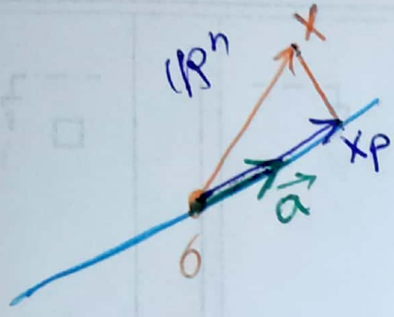
$V\alpha = V(x^T y) = \underbrace{\underset{n\times1}{V}\,\underset{1\times m}{x^T}}_{n\times m}\underset{m\times1}{y}$

LA16 (II)





$x \in \mathbb{R}^n$



$$x - x_p \perp a \Longrightarrow (x - x_p)^T a = 0$$

$$x_p = \alpha \vec{a} = \boxed{\vec{a}\,\alpha}$$

$$(x - x_p)^T a = 0 \Rightarrow x^T\vec{a} = x_p^T \vec{a} = \underbrace{(\alpha\vec{a})^T\vec{a}}$$

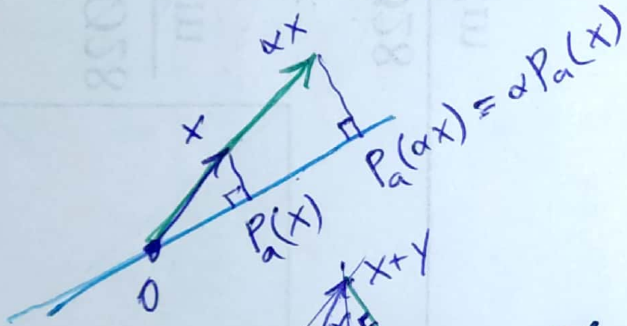
$$\Rightarrow x^T\vec{a} = \alpha\, \vec{a}^T\vec{a} \Rightarrow \alpha = \frac{x^T a}{a^T a} = \frac{\langle x, a\rangle}{\langle a, a\rangle} = \frac{\langle x, a\rangle}{\|a\|^2}$$

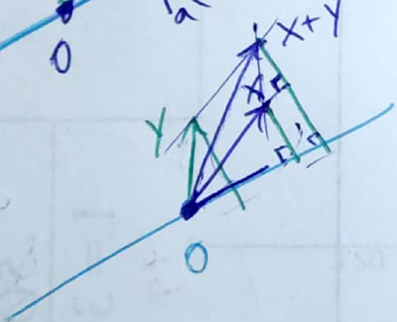
$$x_p = \alpha a = \left(\frac{x^T a}{a^T a}\right)\vec{a}$$

$$x_p = P_a(x) = \left(\frac{x^T a}{a^T a}\right)\vec{a}$$

P_a: projection onto span($\vec{a}$); $\frac{x^T a}{a^T a} \in \mathbb{R}$

$P_a : \mathbb{R}^n \longrightarrow \mathbb{R}^n$





$$P_a(x+y) = P_a(x) + P_a(y)$$

P_a is a linear map $\Rightarrow$ ~~B~~

$$\Rightarrow \boxed{P_a\ \mathbb{R}^n \to \mathbb{R}^n} \quad P_a(x) = \quad x$$

$$\Longrightarrow \exists Q \in \mathbb{R}^{n\times n} \quad P_a(x) = Qx$$

$$P_a(x) = \frac{x^T a}{a^T a}\, a$$

$\alpha \in \mathbb{R}$
$v \in \mathbb{R}^n$

$\underset{1\times 1}{\alpha}\ \underset{n\times 1}{v} \rightarrow$ not a

$\underset{1\times 1}{\alpha}\ \underset{n\times 1}{v} \rightarrow$ not a matrix product

$\underset{n\times 1}{v}\ \underset{1\times 1}{\alpha} \rightarrow$ product of an $n\times 1$ by a 1×1 matrix

$$\alpha = x^T y \qquad x, y \in \mathbb{R}^m$$

$$\alpha v = (x^T y)v \neq x^T \underset{m\times 1}{y}\, \underset{n\times 1}{v}$$

$$v\alpha = v(x^T y) = \underbrace{\underset{n\times 1}{v}\,\underset{1\times m}{x^T}}_{n\times m}\,\underset{m\times 1}{y}$$

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CA 16 (III)

$$P_a(x) = \frac{x^Ta}{a^Ta}\, a = \left(\frac{1}{a^Ta}\right)\underbrace{(x^Ta)}_{1\times1}\underbrace{a}_{n\times1}$$

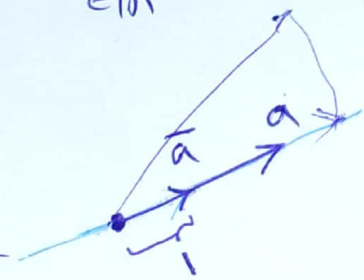
$$= \left(\frac{1}{a^Ta}\right)\underbrace{a}_{n\times1}\underbrace{(x^Ta)}_{1\times1} = \left(\frac{1}{a^Ta}\right)\underbrace{a}_{n\times1}\underbrace{(a^Tx)}_{1\times1}$$

$$\overbrace{\underbrace{n\times1}\quad\underbrace{1\times n}}^{n\times n} \quad = \left(\frac{1}{a^Ta}\right)\underbrace{(\underbrace{a}_{n\times1}\underbrace{a^T}_{1\times n})}_{n\times n} x = \frac{aa^T}{\underbrace{a^Ta}_{\in\mathbb{R}}}\, x$$

$$B = \left(\frac{aa^T}{a^Ta}\right) \in \mathbb{R}^{n\times n} \quad n\times n$$

$$P_a(x) = Bx$$

Bx → projection matrix



$$B = \frac{aa^T}{a^Ta} = \frac{aa^T}{\|a\|^2} = \left(\frac{a}{\|a\|}\right)\left(\frac{a}{\|a\|}\right)^T \qquad \bar a = \frac{a}{\|a\|}$$

$$= \bar a\, \bar a^T \qquad \|\bar a\| = 1 \qquad \bar a^T \bar a = 1$$

$\vec a \neq 0$

$\mathrm{rank}(B) = 1$

$$P_a(x) = \underbrace{\bar a\, \bar a^T}_{n\times n} x$$

$$aa^T = \left[a_1\vec a \quad a_2\vec a \dots a_n\vec a\right]$$

rank $= 1$

$$B = P = \frac{aa^T}{a^Ta}$$

↘ projection matrix

$$P = \left(\frac{1}{a^Ta}\right) aa^T \Rightarrow P^T = \left(\frac{1}{a^Ta}\right) aa^T = P$$

$\Rightarrow P$ symmetric $\boxed{\langle x, Py\rangle = \langle P^Tx, y\rangle}$

$$P_a(P_a(x)) = P(P(x)) = (PP)x = P^2x$$

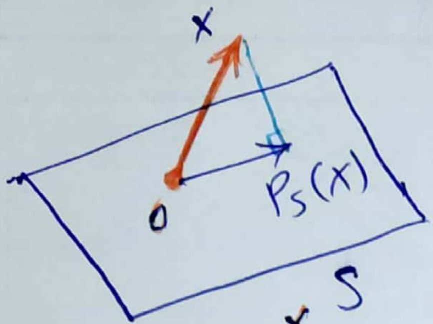
$$P^2 = PP = \left(\frac{1}{a^Ta}\right)(aa^T)\left(\frac{1}{a^Ta}\right)(aa^T) = \left(\frac{1}{a^Ta}\right)^2 \underbrace{(aa^T)}_{n\times n}\underbrace{(aa^T)}_{n\times n}$$

$$= \frac{1}{(a^Ta)^2}\,\underbrace{a}_{m\times1}\,\underbrace{(\underbrace{a^T}_{\cancel{1\times m}}\underbrace{a}_{m\times1})}_{1\times1}\,\underbrace{a^T}_{1\times m} = \frac{(a^Ta)}{(a^Ta)^2}\, aa^T = \frac{1}{a^Ta}\, aa^T$$

$$= P$$

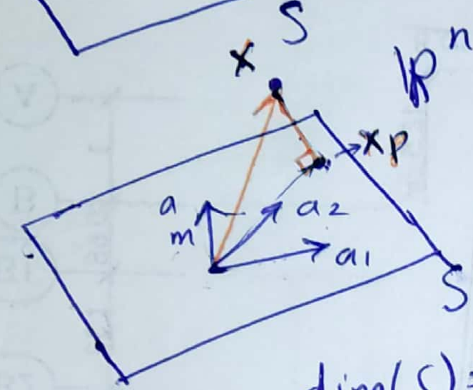
$$P^2 = PP = P \Rightarrow P_a(P_a(x)) = P_a(x) \quad \text{idempotent}$$

LA|6 (IV)



$P_S(x)$

$\vec{a}_1, \vec{a}_2$ $\quad z_i \in \mathbb{R}$



$\mathbb{R}^n$

$$x_p = z_1\vec{a}_1 + z_2\vec{a}_2 + \dots + z_m\vec{a}_m$$

$$x_p = \underbrace{\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_m \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}}_{Z}$$

$\dim(S) = m < n$ $\qquad x_p = Az$

$\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m$ form a basis for S. $\qquad z = \underset{z'}{\operatorname{argmin}} \|Az' - x\|$

$$\Longrightarrow \quad z = (A^TA)^{-1}A^Tx$$

least squares

$$x_p = P_S(x) = Az = \underset{n\times m}{A} \underbrace{(\underset{m\times n}{A^T}\underset{n\times m}{A})^{-1}}_{m\times m} \underset{m\times n}{A^T} x$$

$$P = A(A^TA)^{-1}A^T$$

$P^T = P \rightarrow$ symmetric $\qquad \operatorname{rank}(P) = m = \dim(S)$

$P^2 = P \rightarrow$ idempotent