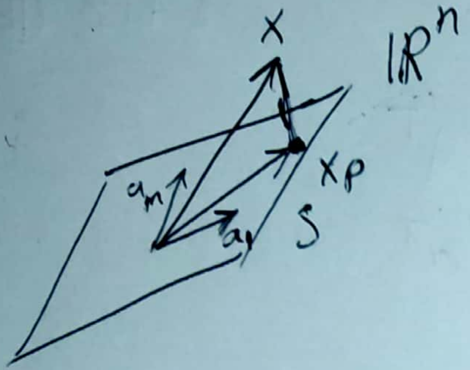


$\in \mathbb{R}^n$

$a_1, a_2, \ldots, a_m$ a basis for S

$\dim(S) = m \quad n \geq m$

$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix}$ has full column rank $\in \mathbb{R}^{n \times m}$

$x_p = P_S(x) = Px$

$P = A(A^TA)^{-1}A^T \qquad P^T = P$

$PP = P^2 = P$

A has full column rank $(\text{rank}(A) = m) \Rightarrow \underline{A^TA \text{ invertible.}}$

~~$B_{p\times p}$ singular ⇒ dependent columns ⇒~~

B non-singular ⇔ independent colums ⇔ $(\cancel{Ax}\ Bx \Rightarrow x = 0)$

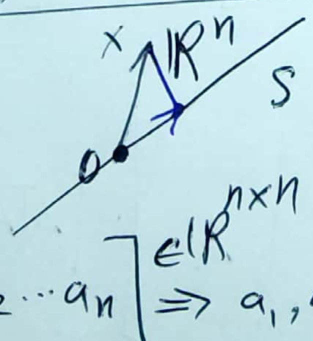
$\underbrace{A^TA}_{m\times m}\underbrace{x}_{m\times 1} = 0 \Rightarrow \underbrace{\underbrace{x^T}_{1\times m}\underbrace{A^TAx}_{m\times 1}}_{1\times 1} = 0 \Rightarrow (\underbrace{Ax}_{\in \mathbb{R}^n})^T(\underbrace{Ax}_{\in \mathbb{R}^n}) = 0$

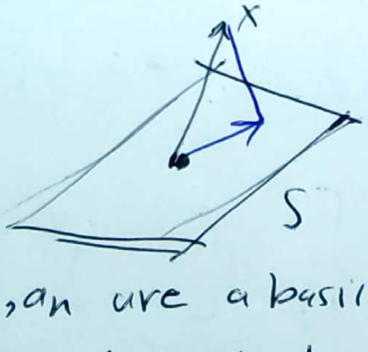
$\Rightarrow \langle Ax, Ax \rangle = \|Ax\|^2 = 0 \Rightarrow Ax = 0 \Rightarrow x = 0$

has full column rank

$\Rightarrow A^TA$ non-singular

$S = \mathbb{R}^n$





$S = \mathbb{R}^n \Rightarrow A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{n\times n} \Rightarrow a_1, a_2, a_n$ are a basis for $\mathbb{R}^n$

$\Rightarrow a_1 - a_n$ idependent $\Rightarrow A$ invertible

$\Rightarrow P = A(A^TA)^{-1}A^T = AA^{-1}(A^T)^{-1}A^T = I \cdot I = I \Rightarrow Px = x$

1D case 

$\dim(S)=1$

$$A = \begin{bmatrix} a \end{bmatrix} \in \mathbb{R}^{n\times 1} \qquad P = A(A^TA)^{-1}A^T$$

$$= a\,\underbrace{(a^Ta)}_{1\times 1}^{-1}a^T$$

$$= \frac{1}{a^Ta}\,a a^T \checkmark$$

1D $\|a\| = \sqrt{a^Ta} = 1 \Rightarrow a^Ta = 1 \Rightarrow P = aa^T$

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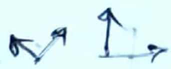
m-D case orthonormal basis

$\to$ basis vectors are perpendicular to each other

$\to$ normal $\|a_i\| = 1$



basis: $a_1, a_2, \dots, a_m$

$$\begin{cases} \|a_i\| = 1 \text{ for all } i = 1,\dots,m \Rightarrow a_i^Ta_i = 1 \\ a_i \perp a_j \text{ for all } i \neq j \Rightarrow a_i^Ta_j = 0 \quad i \neq j \end{cases}$$

$$\boxed{a_i^Ta_j = \delta_{ij}}$$

$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix}$ has orthonormal columns

$A^TA = ?$ $\quad A^TA = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}\begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_{m\times m}$

$(A^TA)_{ij} = a_i^Ta_j$

$$\begin{bmatrix} A^T \end{bmatrix}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} A^TA \end{bmatrix} = \begin{bmatrix} I \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} A^T \end{bmatrix}^{n\times m} = \begin{bmatrix} \underset{n\times n}{AA^T} \end{bmatrix} \neq I \text{ if } m < n$$

A has orthonormal columns

$A \in \mathbb{R}^{n\times m}$ $\quad A = \begin{bmatrix} \quad \end{bmatrix} \Rightarrow A^TA = I$

AA^T

Case 1: $m < n \Rightarrow \operatorname{rank}(AA^T) < n \Rightarrow AA^T \neq I$

$AA^T \in \mathbb{R}^{n\times n}$ $\qquad$ $\operatorname{rank}(I) = n$

Case 2: $m = n \Rightarrow A \in \mathbb{R}^{n\times n}$ has n idependent columns

A: non-singular / invertible

$$\left.\begin{array}{l} A^TA = I \\ A \text{ non-singular (square)} \end{array}\right\} \Rightarrow A^T = A^{-1} \Rightarrow A^TA = AA^T = I$$

$$\boxed{AA^T = I}$$

$\Rightarrow A$ has orthonormal rows

A square with orthonormal columns

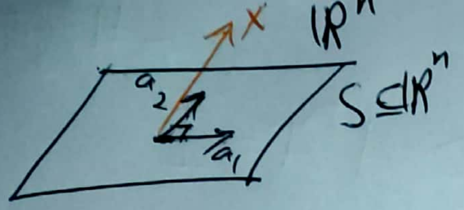
$\Rightarrow A^TA = AA^T = I \quad (A^{-1} = A^T)$

$\Rightarrow A$ also has orthonormal rows.

A is called an orthogonal matrix

Example: Permutation matrix $P = P^{-1}$ $\quad PP^T = I$ $\quad P^TP = I$

Rotation matrix R $\quad R^TR = I$



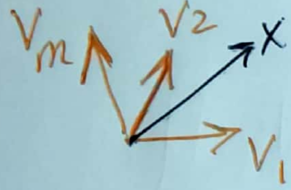
$S \subseteq \mathbb{R}^n$ $a_1, a_2, \ldots, a_m$ form an orthonormal basis for S: $\underline{A^TA = I}$

$A = [a_1 — a_m]$

$$\underline{P} = A(A^TA)^{-1}A^T = A(I)^{-1}A^T = AA^T$$

$$P = AA^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} = \underbrace{a_1a_1^T}_{n\times n} + \underbrace{a_2a_2^T}_{n\times n} + \cdots + a_ma_m^T$$

$$Px = a_1\underline{a_1^Tx} + a_2\underline{a_2^Tx} + \cdots + a_m a_m^T x$$



$x \in \mathbb{R}^m$ what ~~is~~ are the coordinates of x in basis $v_1, v_2, \ldots, v_m \in \mathbb{R}^m$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$ → coordinates of x in $v_1 —— v_m$

$$x = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_m\vec{v_m} = \underbrace{\begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_m} \end{bmatrix}}_{\substack{m\times m \\ \text{invertible}}} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = x$$

$$\vec{a} = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_m} \end{bmatrix}^{-1} x$$

what if $v_1, v_2 \cdots v_m$ are orthonormal.

$$\begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_m} \end{bmatrix}^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}^T = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix} \Rightarrow \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix}$$

$$\Rightarrow \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_m^T \end{bmatrix} x = \begin{bmatrix} v_1^Tx \\ v_2^Tx \\ \vdots \\ v_m^Tx \end{bmatrix} \Rightarrow a_i = v_i^T x$$