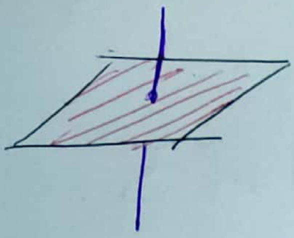


$\underline{A \in \mathbb{R}^{m \times n}}$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} & A & \end{bmatrix} \end{array} \begin{bmatrix} x \end{bmatrix}$$

$C(A) \subseteq \mathbb{R}^m$

$R(A) = C(A^T) \subseteq \mathbb{R}^n$

$N(A) = \{x \mid Ax = 0\} \subseteq \mathbb{R}^n$

left null space $\leftarrow$ $N(A^T) = \{x \mid x^T A = \vec{0}^T\} \subseteq \mathbb{R}^m$

$A^T x = \vec{0}$

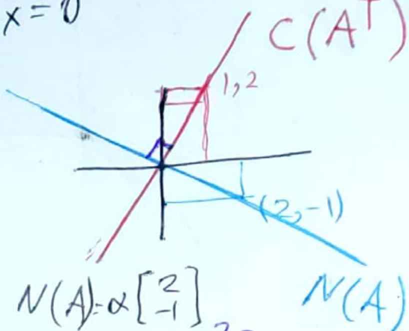
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$C(A) \subseteq \mathbb{R}^3$

$C(A^T) \subseteq \mathbb{R}^2$

$N(A) \subseteq \mathbb{R}^2$

$N(A^T) \subseteq \mathbb{R}^3$



$N(A) = \alpha \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

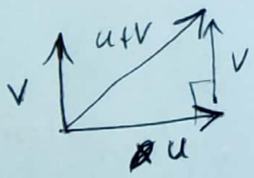
$x \in C(A^T) \Rightarrow x = A^T z$

$y \in N(A) \Rightarrow Ay = 0$

$\langle x, y \rangle = x^T y = (A^T z)^T y = z^T A y = z^T \vec{0} = 0$

$\Rightarrow x \perp y \xrightarrow[x,y \text{ arbitrary}]{} C(A^T) \perp N(A)$

$\Rightarrow C(A) \perp N(A^T)$



$\|u+v\|^2 = \|u\|^2 + \|v\|^2$

$(u+v)^T(u+v) = u^T u + v^T v \Rightarrow u^T v = 0$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} & A & \end{bmatrix} \end{array}$$

$\dim(C(A)) = r$

$\dim(C(A^T)) = r$

$\dim(N(A)) = n - r = n - \dim(C(A^T))$

$\dim(N(A^T)) = m - r = m - \dim(C(A))$

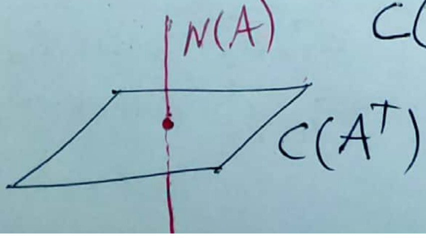
$A \in \mathbb{R}^{m \times n}$

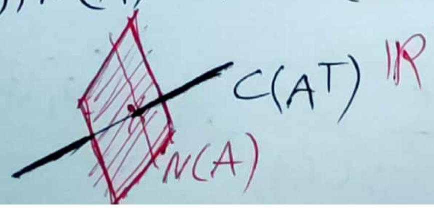
$\text{Rank}(A) \doteq r$

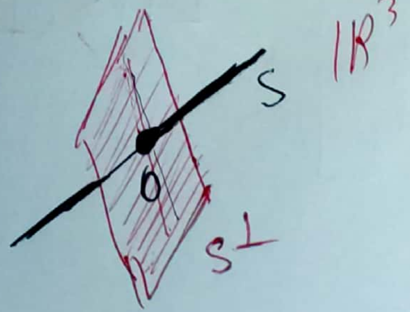
$C(A), N(A^T) \subseteq \mathbb{R}^m \quad \dim(C(A)) + \dim(N(A^T)) = m$

$C(A^T), N(A) \subseteq \mathbb{R}^n \quad \dim(C(A^T)) + \dim(N(A)) = n$

$n = 3$



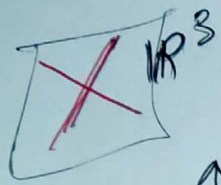




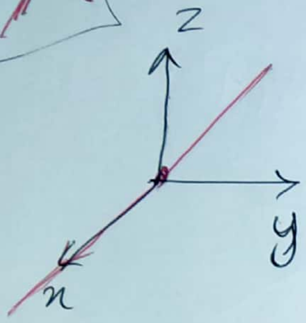
S : a linear subspace

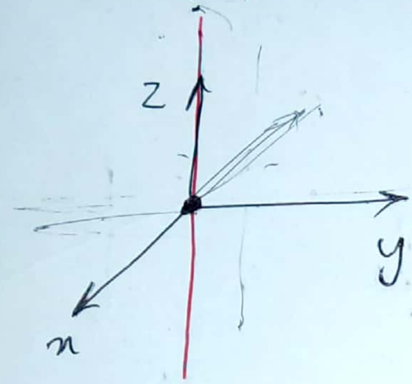
$$S^\perp = \{x \mid x \perp y \ \forall y \in S\}$$

→ orthogonal complement of S.

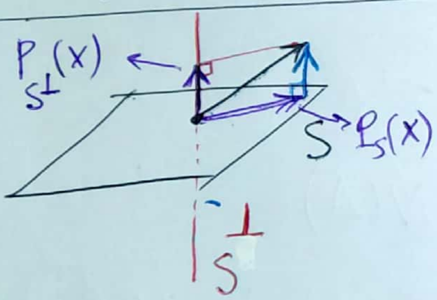


$$C(A^T)^\perp = N(A)$$
$$C(A)^\perp = N(A^T)$$





$$(S^\perp)^\perp = S$$



$$P_S(x) = Px = A(A^TA)^{-1}A^T x$$

$$P_{S^\perp}(x) = x - Px = (I-P)x$$
$$= x - P_S(x)$$

$(I-P)^T = I^T - P^T = I - P$ symmetric

$$(I-P)^2 = (I-P)(I-P) = I^2 - IP - PI + P^2$$
$$= I - P - P + P = I - P$$ → idempotent

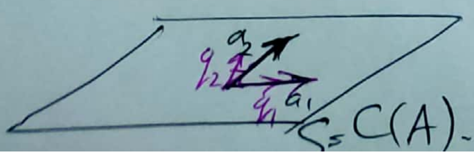
P : Projection matrix onto S.

I-P : 〃 〃 onto $S^\perp$

Columns of A form a basis for S.

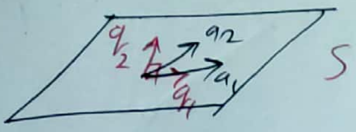
$\Longrightarrow S = C(A)$

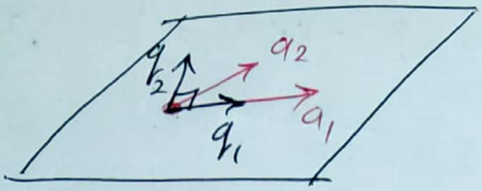
$S^\perp = N(A^T)$



$\vec{a}_1, \vec{a}_2$ basis for S

$A = [\vec{a}_1 \ \vec{a}_2]$ $\Longrightarrow \vec{q}_1, \vec{q}_2$ orthonormal basis for S





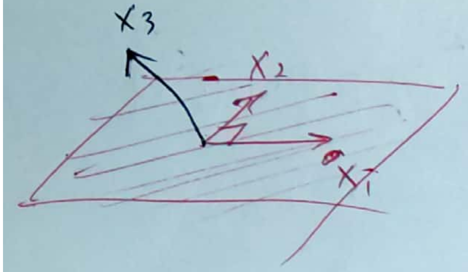
find q_1, q_2 s.t. LA18 (III)

$span(q_1, q_2) = span(a_1, a_2)$

q_1, q_2 orthonormal

$\hookrightarrow q_1^T q_1 = q_2^T q_2 = 1$

$q_1^T q_2 = 0$



$S = C(A)$

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{m \times n} \quad Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$m \geq n$

full column rank

Given $A \in \mathbb{R}^{m \times n}$ find $Q \in \mathbb{R}^{m \times n}$ with orthonormal columns $(Q^T Q = I)$ such that $C(Q) = C(A)$.

$span(q_1, q_2, \ldots, q_n) = span(a_1, \ldots a_n)$

(orthogonalization)

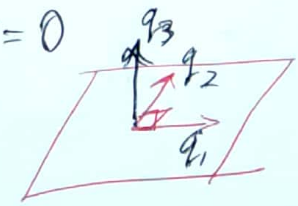
$S = C(A)$

$$P_S(x) = \underbrace{A(A^TA)^{-1}A^T}_{P} x = \underbrace{QQ^T}_{P} x = (q_1 q_1^T + q_2 q_2^T + \cdots + q_n q_n^T) x$$

$$= q_1 (q_1^T x) + q_2 (q_2^T x) + \cdots + q_n (q_n^T x)$$

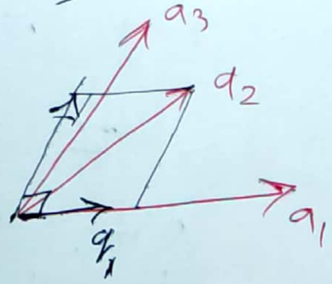
$$P_{S^\perp}(x) = (I - QQ^T) x = x - q_1 q_1^T x - q_2 q_2^T x - \cdots + q_n q_n^T x$$

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \quad Q^\perp = \underbrace{\begin{bmatrix} q_{n+1} & q_{n+2} & \cdots & q_m \end{bmatrix}}_{\text{basis for } S^\perp} \quad Q^T Q^\perp = 0$$



$$P_{S^\perp}(x) = (I - QQ^T) x = Q^\perp (Q^\perp)^T x$$

$$A \longrightarrow Q \qquad \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$$



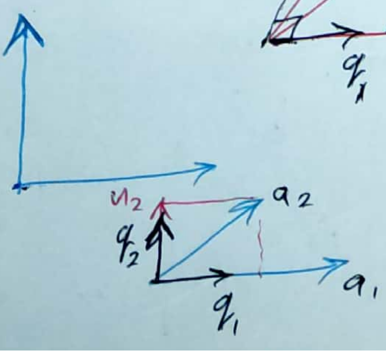
$$q_1 = \frac{a_1}{\|a_1\|}$$

$$u_2 = a_2 - q_1 q_1^T a_2$$

$$q_2 = \frac{u_2}{\|u_2\|}$$

$$u_3 = a_3 - q_1 q_1^T a_3 - q_2 q_2^T a_3$$

$$q_3 = \frac{u_3}{\|u_3\|}$$



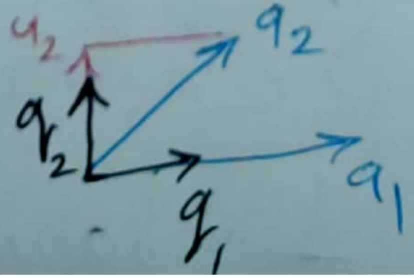
LA 18

IV

$$\left.\begin{aligned} u_1 &= a_1 \\ q_1 &= \frac{u_1}{\|u_1\|} \end{aligned}\right\} \quad \operatorname{span}(q_1) = \operatorname{span}(a_1)$$

$$q_1^T q_1 = 1$$

$$\left.\begin{aligned} u_2 &= a_2 - q_1 q_1^T a_2 \\ q_2 &= \frac{u_2}{\|u_2\|} \end{aligned}\right\} \quad \operatorname{span}(q_1, q_2) = \operatorname{span}(a_1, a_2)$$



$$[q_1, q_2]^T [q_1, q_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$