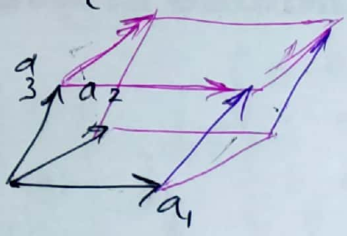
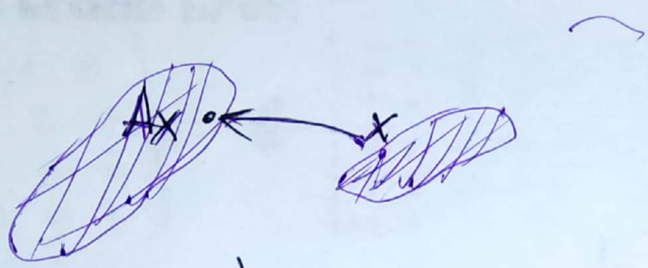
Signed Volume $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \in \mathbb{R}^n$

$$A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n] \in \mathbb{R}^{n\times n}$$





$$V(A) = V(a_1, a_2, \dots, a_n)$$

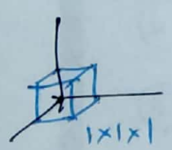
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \vec{a}_1 = \begin{bmatrix} a \\ c \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} b \\ d \end{bmatrix} \quad V(A) = V\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right) =$$

$$V\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - cb$$

$$= \det(A)$$

$$|A| = \det(A) = V(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n) \quad \left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\det(A) = \det([\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n]) = |\vec{a}_1 \; \vec{a}_2 \; \dots \; \vec{a}_n|$$



$$\det(I) = V(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n) = 1$$

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Multilinear / linear in each column

$$\det([a_1, a_2, \alpha a_3 + \beta a_3', a_4]) = \alpha \det([a_1\ a_2\ a_3\ a_4]) + \beta \det[a_1\ a_2\ a_3'\ a_4]$$

$$\det[a_1\ a_2\ a_2\ a_4] = 0$$

$$\det([u, v, \alpha w + \beta t, z]) = \alpha \det[u, v, w, z] + \beta \det[u, v, t, z]$$

$\forall\, u, v, w, t, z \in \mathbb{R}^4$

LA20 (II)

$$\begin{bmatrix} x & y & z & t \end{bmatrix} = - \begin{bmatrix} x & z & y & t \end{bmatrix}$$

$$|P| \quad \det \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1$$

$$\det(P) = \begin{cases} 1 & P \text{ is an } \underline{\text{even}} \text{ permutation} \\ -1 & P \text{ is an } \underline{\text{odd}} \text{ permutation} \end{cases}$$

$$\left| a_1 \ a_2 \ \vec{0} \ a_4 \right| = \left| \vec{a}_1 \ \vec{a}_2 \ 0\cdot\vec{x} \ a_4 \right| = 0 \cdot \left| \vec{a}_1 \ \vec{a}_2 \ \vec{x} \ \vec{a}_4 \right| = 0$$

$$\left| a_1 \ a_2 \ x - x \ a_4 \right| =$$

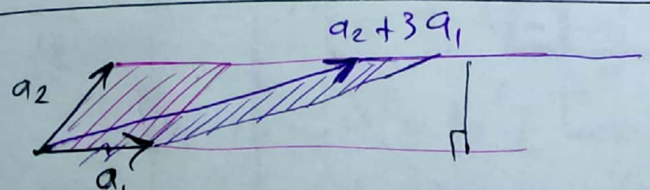
$$\left| \vec{a}_1 \ \vec{a}_2 \ \vec{0} \ \vec{a}_4 \right| = - \left| \vec{a}_1 \ \vec{a}_2 \ -\vec{0} \ \vec{a}_4 \right| = - \left| \vec{a}_1 \ \vec{a}_2 \ \vec{0} \ a_4 \right|$$

~~det~~ $A \in \mathbb{R}^{n \times n}$ $\quad \alpha \in \mathbb{R}$

$$\det(\alpha A) = \det \begin{bmatrix} \alpha a_1 \ \alpha a_2 \ \cdots \ \alpha a_n \end{bmatrix} = \alpha^n \det(A)$$

~~det~~ $\left| a_1 \ a_2 \ \alpha a_1 + \beta a_2 \right| = \alpha \left| a_1 \ a_2 \ a_1 \right| + \beta \left| a_1 \ a_2 \ a_2 \right|$

$a_1, a_2 \in \mathbb{R}^3$ $\quad = \alpha \cdot 0 + \beta \cdot 0 = 0$



$$\det(-4) = -4 \det(1) = -4$$

$|x|$ $\quad$ I_1

triangular matrix

$c_2 \mathrel{-}= 7c_1$, $c_3 \mathrel{+}= 2c_1$, $c_3 \mathrel{+}= 2c_2$

$$\begin{vmatrix}1&7&-2\\0&2&-4\\0&0&3\end{vmatrix} = \begin{vmatrix}1&0&0\\0&2&-4\\0&0&3\end{vmatrix} = \begin{vmatrix}1&0&0\\0&2&0\\0&0&3\end{vmatrix} = 1\times2\times3$$

$A \in \mathbb{R}^{n\times n}$ Upper/Lower triangular $\Rightarrow |A| = \prod_{i=n}^{n} a_{ii}$

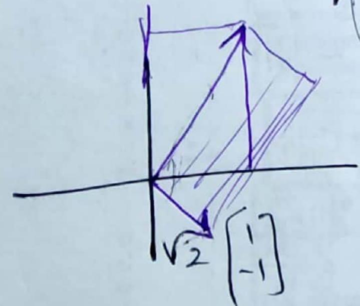
$\det(AB) = \det(A)\det(B)$

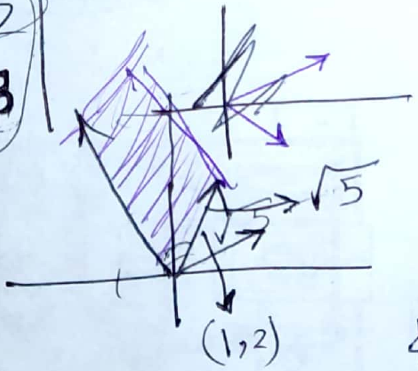
$\det(A^{-1})$ $\quad AA^{-1} = I \Rightarrow \det(AA^{-1}) = \det(I)$

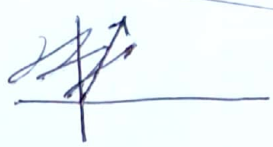
$\Rightarrow \det(A)\det(A^{-1}) = 1$

$$|A^{-1}| = \frac{1}{|A|}$$

$|A^T|$ $\quad \begin{vmatrix}1&2\\-1&3\end{vmatrix}$







$|P^{-1}| = |P^T| = \frac{1}{|P|} = |P|$

$|P| \in \{-1, 1\}$

LU decomposition $\quad A$

permutation $\leftarrow$ $PA = LU \Rightarrow |P||A| = |L||U| \Rightarrow$

$\cancel{P}A^TP^T = U^TL^T \Rightarrow |\cancel{P}A^T||P^T| = |U^T||L^T| \Rightarrow$

$|A| = |L||U|/|P| = \left(\prod_{i=1}^{n} l_{ii}\right)\left(\prod_{i=1}^{n} u_{ii}\right)/|P|$

$|A^T| = |L^T||U^T|/|P^T| = \left(\prod l_{ii}\right)\left(\prod u_{ii}\right)/|P|$

$\Rightarrow |A| = |A^T|$

Scanned with CamScanner

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\begin{vmatrix} 1 & b \\ 0 & d \end{vmatrix} + c\begin{vmatrix} 0 & b \\ 1 & d \end{vmatrix}$$

$$= \cancel{ab\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}} + ad\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \cancel{cd\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \underbrace{\cancel{\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix}}}_{0} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \underbrace{\cancel{\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix}}}_{0}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & & \\ 0 & & \\ 0 & & \end{vmatrix} + \begin{vmatrix} & & \\ & & \\ & & \end{vmatrix} + \begin{vmatrix} & & \\ & & \\ & & \end{vmatrix}$$

$$A A^T = C \underbrace{R R^T}_{\text{invertible}} \underbrace{C^T}_{\text{full row rank}}$$

$x = C^T y$ $\qquad R R^T C^T y$