$y = Ax \quad A \in \mathbb{R}^{n \times n} \quad x \in \mathbb{R}^n$

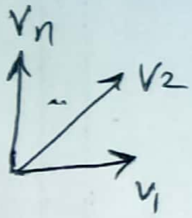
$\Rightarrow O(n^2)$ operation in general.

$$A = \begin{bmatrix} a_1 & & & \emptyset \\ & a_2 & & \\ & & a_3 & \\ \emptyset & & & \ddots\; a_n \end{bmatrix} \qquad Ax = \begin{bmatrix} a_1 & & \emptyset \\ & a_2 & \\ \emptyset & & \ddots\; a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \\ \vdots \\ a_n x_n \end{bmatrix}$$

A diagonal $\quad y = Ax$ needs $O(n)$ operation

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$x \in \mathbb{R}^n$

$x = z_1 \vec{v}_1 + z_2 \vec{v}_2 + \dots + z_n \vec{v}_n$

$v_1, v_2 \dots v_n$ a basis for $\mathbb{R}^n$

$V = [v_1, v_2, \dots, v_n] \in \mathbb{R}^{n \times n}$

$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \rightarrow$ coordinates of x in the basis formed by $v_1, v_2 \dots v_n$

$$\vec{x} = [v_1\ v_2 \dots v_n] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = V\vec{z}$$

$$\vec{z} = V^{-1}\vec{x}$$

choose $v_1, v_2, \dots v_n$ such that the transformation A is diagonal in this new basis.

Let D be the diagonal matrix performing the same transformation as A in the new basis.

$x' = Ax$

$z' = Dz, \ z = V^{-1}x, \ z' = V^{-1}x' \Rightarrow V^{-1}x' = DV^{-1}x \Rightarrow V^{-1}Ax = DV^{-1}x$

$$Ax = VDV^{-1}x \Rightarrow \boxed{A = VDV^{-1}}$$

$$\boxed{A = VDV^{-1}} \qquad D = V^{-1} A V$$

If there exist such a matrix V we say that A is diagonalizable.

$A, B \in \mathbb{R}^{n\times n}$ are similar if $\exists\, V \in \mathbb{R}^{n\times n}$ such that

$$A = VBV^{-1}$$

A diagonalizable $\iff$ A is similar to a diagonal matrix.

$\cancel{A = [v_1\, v_2 \cdots v_n]}$

$V = [v_1, v_2, \cdots v_n] \in \mathbb{R}^{n\times n}$ such that $A = VDV^{-1}$

$Av_1 = ?$ $\quad V^{-1} v_1 = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad V^{-1} v_i = e_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow i$

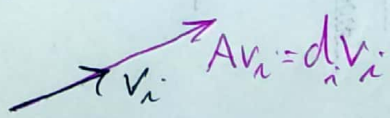
$$V^{-1} V = V^{-1} [v_1\, v_2 \cdots v_n] = I = [e_1\, e_2 \cdots e_n]$$

$$Av_1 = VDV^{-1} v_1 = VDe_1 = [v_1\, v_2 \cdots v_n] \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= [v_1\, v_2 - v_n] \begin{bmatrix} d_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= d_1 \vec{v}_1$$

$$A\vec{v}_i = d_i \vec{v}_i$$



Assume that we have found such a basis. $v_1, v_2 - v_n$.

$$Ax = VDV^{-1}x \qquad A^2 = AA = VDV^{-1}VDV^{-1} = VD^2V^{-1}$$

Scanned with CamScanner

$$A = VDV^{-1}$$

$$D = \begin{bmatrix} d_1 & & & \emptyset \\ & d_2 & & \\ & & \ddots & \\ \emptyset & & & d_n \end{bmatrix} \qquad \underline{\underline{A^2 = VD^2V^{-1}}}$$

$$D^2 = DD = \begin{bmatrix} d_1^2 & & & \emptyset \\ & d_2^2 & & \\ & & \ddots & \\ \emptyset & & & d_n^2 \end{bmatrix}$$

$$\underline{\underline{A^n}} = V D^n V^{-1}$$

$O(n^4)$ $\quad \searrow O(n^2)$

$O(n^3 \log n)$

$A, B \in \mathbb{R}^{n\times n}$ are jointly diagonalizable if $\exists V \in \mathbb{R}^{n\times n}$

$$A = V D_1 V^{-1} \Rightarrow D_1 = V^{-1} A V$$

$$B = V D_2 V^{-1} \qquad D_2 = V^{-1} B V$$

Ex: A, B circulant matrices

$$Ax = \begin{bmatrix} a & b & c & 0 & 0 \\ 0 & a & b & c & 0 \\ 0 & 0 & a & b & c \\ c & 0 & 0 & a & b \\ b & c & 0 & 0 & a \end{bmatrix} x$$

circular convolution

Fourier basis

$$Ax = V D_A \boxed{V^{-1}}$$

$$BA = V D_2 V^{-1} V D_1 V^{-1} = V D_2 D_1 V^{-1}$$

$$= V D_1 D_2 V^{-1}$$

$$= V D_1 V^{-1} V D_2 V^{-1} = AB$$

$BA = AB$

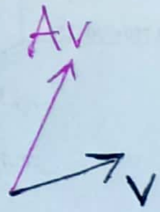
$\underline{\underline{A}}\, \underline{\underline{v_i}} = d_i\, \underline{\underline{v_i}}$ LA[22] (IV)

Let $A \in \mathbb{R}^{n \times n}$, $\vec{v} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ be such that

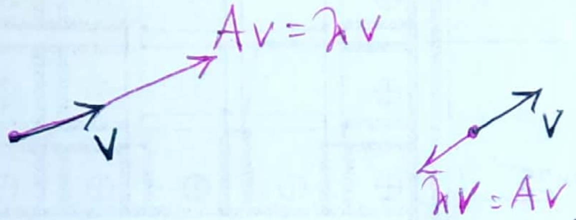
$v \neq \emptyset$

$$A\vec{v} = \lambda \vec{v}$$

Eigenvector — Eigenvalue



v not an eigen~~value~~ vector of A



v is an eigen~~value~~ vector of A

Example A diagonal

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{eigenvector}} = \underbrace{4}_{\text{eigenvalue}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underbrace{\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 3\right)}_{\text{eigen pair}}, \underbrace{\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 4\right)}_{\text{eigenpair}}$$

. Example : Shear

$$A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \quad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 1 \right)$$

Find all eigenvectors of A $(\alpha \neq 0)$

$$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 + \alpha v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix} \quad \text{I}$$

$$v_2 = \lambda v_2 \begin{cases} v_2 = 0 \\ \lambda = 1 \end{cases} \Rightarrow \left.\begin{cases} v_1 + \alpha v_2 = v_1 = \lambda v_1 \\ v_1 \neq 0 \end{cases}\right\} \Rightarrow \lambda = 1$$

$$\lambda = 1 \underset{\text{I}}{\Rightarrow} \begin{bmatrix} v_1 + \alpha v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \overset{\alpha \neq 0}{\Rightarrow} v_2 = 0$$

eigenvalue $= 1$

eigenvector $= \begin{bmatrix} v_1 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ → only 1 eigenvector!

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 1 \right)$$

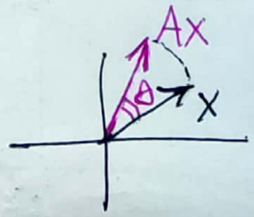
Example: the identity matrix

$$I x = 1 \cdot x$$

Every $x \in \mathbb{R}^n$ is an eigenvector of I whose corresponding eigenvalue is 1.

Example: 2D Rotation $\theta \neq k\pi$ $\quad Ax = \alpha x$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

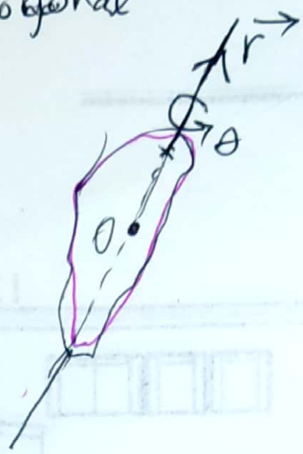


No real Eigenvectors

3D Rotation

$\underset{3\times 3}{\underbrace{A}}x$ $\quad$ A ~~o~~ orthogonal $(A^TA = AA^T = I)$

$|A| = 1$



$A\vec{r} = 1\vec{r}$

axis of rotation

Singular

A singular $\Rightarrow \exists\, u \in \mathbb{R}^n$, $u \neq 0$, $Au = 0$

null vector

$Au = 0.u$

eigenvector $\quad$ eigenvalue

Every ~~n~~ (non-zero) null-vector of A is an eigenvector. (eigenvalue $= 0$)