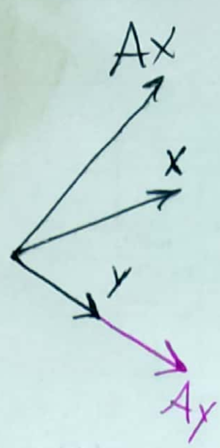
$A \in \mathbb{R}^{n \times n}$

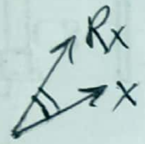
$Av = \lambda v$

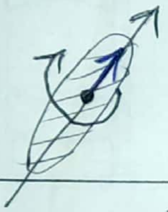


Diagonalization $\mathbb{R}^n$ $V = [v_1 \; v_2 \cdots v_n]$

$A = VDV^{-1}$

$Av_i = d_i \cdot v_i$





Shear $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$

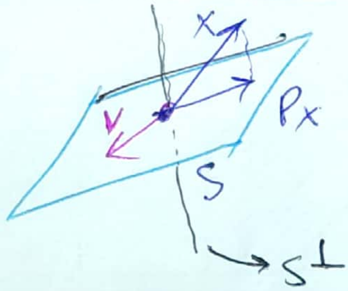
Singulare $\Rightarrow \exists v \neq 0 \in \mathbb{R}^n$ $Av = 0$

$Av = 0 \cdot v$

Projection Matrix

$P = P^T$

$PP = P$



$\forall v \in S$ $\quad Pv = v = 1 \cdot v$

eigenvector eigenvalue

$\forall v \in S^\perp$ $\quad Pv = 0 = 0 \cdot v$

Let $\vec{v}$ be an eigenvector of $P \Rightarrow Pv = \lambda v$

$$\left.\begin{aligned} \Rightarrow PPv = \lambda Pv = \lambda^2 v \\ (PP)v = Pv = \lambda v \end{aligned}\right\} \left.\begin{aligned} \lambda^2 v = \lambda v \\ v \neq 0 \end{aligned}\right\} \lambda^2 = \lambda \Rightarrow \begin{cases} \lambda = 0 \\ \lambda = 1 \end{cases}$$

$A \in \mathbb{R}^{n\times n}$

Let (v,λ) be an eigenpair of A.

$$A v = \lambda v \Rightarrow \underbrace{A}_{n\times n} v - \underbrace{\lambda}_{1\times 1} v = 0 \Rightarrow \underbrace{A}_{n\times n} v - (\underbrace{\lambda I}_{n\times n}) v = 0$$

$$\left.\begin{array}{l}(A-\lambda I)v = 0 \\ v \neq 0\end{array}\right\} \Rightarrow (A-\lambda I) \text{ singular}$$

$$\Rightarrow \det(A-\lambda I) = 0$$

Converse:

$$\det(A-\lambda I)=0 \Rightarrow (A-\lambda I) \text{ singular}$$

$$\Rightarrow \exists v\in\mathbb{R}^n,\ v\neq 0 \quad (A-\lambda I)v=0 \Rightarrow Av=\lambda v$$

$\Rightarrow \lambda$ is an eigenvalue of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow A-\lambda I = A - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) - (a_{11}-\lambda)a_{32}a_{23} - a_{21}a_{12}(a_{33}-\lambda) \pm \cdots$$

$$\Rightarrow -\lambda^3 + \cdots + a\lambda + b$$

$\Rightarrow$ a polynomial of degree $3 \Rightarrow$ at most 3 roots

$\Rightarrow$ at most 3 eigenvalue.

$A \in \mathbb{R}^{n \times n}$

$\det(A - \lambda I)$ is a polynomial of degree $\underline{\underline{n}}$. (in λ)

$p(\lambda) = \det(A - \underline{\lambda} I)$

$\Longrightarrow$ characteristic Polynomial

$\Longrightarrow$ A matrix $A \in \mathbb{R}^{n \times n}$ has at most $\underline{\underline{n}}$ eigenvalues.

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)^2 - 3^2 = (\lambda^2 - 4\lambda + 4) - 9 = 0$$

$$\Longrightarrow \lambda^2 - 4\lambda - 5 = 0 \Rightarrow (\lambda - 5)(\lambda + 1) = 0$$

$$\boxed{\lambda_1 = 5}, \quad \boxed{\lambda_2 = -1}$$

$$A - \lambda_1 I = \begin{bmatrix} 2-5 & 3 \\ 3 & 2-5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \Longrightarrow \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 5 \right)$$

$$A - \lambda_2 I = \begin{bmatrix} 2+1 & 3 \\ 3 & 2+1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \Longrightarrow \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, -1 \right)$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ Sometimes we are interested in normalized eigen vectors ($\|v\| = 1$)

$$\left(\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, 5 \right)$$

$$\left(\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}, -1 \right)$$

LA23 IV

$$A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \qquad |A - \lambda I| = \begin{vmatrix} -\lambda & 3 \\ 3 & -\lambda \end{vmatrix} = \lambda^2 - 3^2 = 0$$

$$\lambda_1 = +3$$
$$\lambda_2 = -3$$

$$A - \lambda_1 I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \implies v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \implies v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \qquad \alpha \neq 0$$

$$\det(A - \lambda I) = \left| \begin{bmatrix} 1-\lambda & \alpha \\ 0 & 1-\lambda \end{bmatrix} \right| = (1-\lambda)^2 = 0$$

repeated root

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$\lambda = 1$

$\to$ algebraic multiplicity $= 2$

$$(A - \lambda I)v = 0 \implies \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \cancel{v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}}$$

$v \neq 0$

$$\xRightarrow{\alpha \neq 0} v_2 = 0 \implies \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{v} \neq 0 \implies v_1 \neq 0$$

Rotation $\quad R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \theta = \frac{\pi}{2} \implies R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\det(R - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0$$
$$\lambda = \pm i$$

$$\lambda_1 = i \implies (R - \lambda_1 I)v = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \implies v_1 = \begin{bmatrix} -1 \\ i \end{bmatrix} \qquad \left(i, \begin{bmatrix} -1 \\ i \end{bmatrix}\right)$$

$$\lambda_2 = -i \implies (R - \lambda_2 I)v = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} v = 0 \implies v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \left(-i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right)$$

$A \in \mathbb{C}^n$

$\det(A - \lambda I)$: a polynomial of degree n.

$\Rightarrow$ at most $\underline{n}$ roots

$\Rightarrow$ exactly n roots (counting the repeated roots)

Assume that the eigenvectors v_1, v_2 of $A \in \mathbb{C}^{n\times n}$ share an eigenvalue.

$$\Rightarrow \begin{aligned} Av_1 &= \lambda v_1 \\ Av_2 &= \lambda v_2 \end{aligned}$$

What can we say about their linear combination?

$$A(\alpha v_1 + \beta v_2) = \alpha A v_1 + \beta A v_2 = \alpha \lambda v_1 + \beta \lambda v_2$$
$$= \lambda(\alpha v_1 + \beta v_2)$$

$\alpha v_1 + \beta v_2$ is also an eigenvector of A.

Let $\underline{\lambda}$ be an eigenvalue of $A \in \mathbb{C}^{n\times n}$

The set of all eigenvectors corresponding to λ is a $\underline{\text{linear}}$ subspace.

$$V(\lambda) = \{v \mid Av = \lambda v\} = \{v \mid [A - \lambda I] \cdot v = 0\} = N(A - \lambda I)$$

a linear subspace

Eigenspace of λ.