$V(\lambda)=\{v \mid Av=\lambda v\} = N(A-\lambda I)$ is a linear subspace

↳ eigenspace corresponding to λ

$A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\quad \forall v\in\mathbb{R} \quad Av = v = 1\cdot v \quad \lambda=1$

eigenspace$(1) = \mathbb{R}^2$

$V(1)=\mathbb{R}^2$ $\quad$ dim(eigenspace) = geometric multiplicity

$|A-\lambda I| = \begin{bmatrix}1-\lambda&0\\0&1-\lambda\end{bmatrix} \Rightarrow (1-\lambda)^2=0 \Rightarrow$ algebraic multiplicity

$\lambda=1$ repeated

for $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ for $\lambda=1$ $\begin{cases}\text{algebraic multiplicity}=2\\ \text{geometric} \quad '' \quad =2\end{cases}$

$A=\begin{bmatrix}1&\alpha\\0&1\end{bmatrix}$ $\quad$ eigenspace $\beta\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \dim=1$

$\lambda=1$

$\alpha\neq 0 \Rightarrow \begin{cases}\text{geometric mult.}=1\\ \text{algebraic} \quad '' \quad =2\end{cases}$

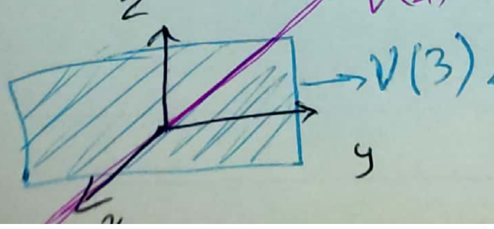
$A=\begin{bmatrix}2&0\\0&3\end{bmatrix}$ $\quad \left(2, \alpha\begin{bmatrix}1\\0\end{bmatrix}\right)$ $\quad \lambda=2 \quad V(\lambda)=\left\{\alpha\begin{bmatrix}1\\0\end{bmatrix} \mid \alpha\in\mathbb{R}\right\}$

$\lambda=3 \quad V(\lambda)=\left\{\alpha\begin{bmatrix}0\\1\end{bmatrix} \mid \alpha\in\mathbb{R}\right\}$

$A=\begin{bmatrix}2&0&0\\0&3&0\\0&0&3\end{bmatrix}$ $\quad \lambda=2 \Rightarrow V(\lambda)=\left\{\alpha\begin{bmatrix}1\\0\\0\end{bmatrix} \mid \alpha\in\mathbb{R}\right\}$

geometric mult = algeb. mult. = 1

$\lambda=3 \Rightarrow V(\lambda)=\left\{\alpha\begin{bmatrix}0\\1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\0\\1\end{bmatrix} \mid \alpha,\beta\in\mathbb{R}\right\}$



2 = geo. mult = alg mult

LA24 (II)

Rotation $\theta = 90$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\lambda = \pm i$ $v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

real complex

$A \in \mathbb{R}^{n\times n}$ $\det(A-\lambda I)$: a polynomial of degree $\underline{n}$ on $\underline{\lambda}$.

$\hookrightarrow$ has exactly $\underline{\underline{n}}$ roots (counting the repeated ones)

complex

$\sum$ algebric multiplicity $= n$

Complex matrices $\underset{\sim}{A} \in \mathbb{C}^{n\times n}$

Same thing is true about $\det(A-\lambda I)$

$c = \underbrace{a}_{\text{real part}} + i\underbrace{b}_{\text{imaginary part}} \in \mathbb{C}$ $re^{i\theta}$

$\bar{c} = a - ib \in \mathbb{C}$ $re^{-i\theta}$

مزدوج
conjugate

$u = \begin{bmatrix} 2i \\ -4+3i \\ 4 \end{bmatrix} \in \mathbb{C}^3$ $\bar{u} = \begin{bmatrix} -2i \\ -4-3i \\ 4 \end{bmatrix} \in \mathbb{C}^3$ $u \in \mathbb{R}^n$ $\bar{u} \in \mathbb{R}^n$

$A = \begin{bmatrix} i & 3+i \\ -2i & 4 \\ i+2 & 6-i \end{bmatrix}$ $\bar{A} = \begin{bmatrix} -i & 3-i \\ 2i & 4 \\ -i+2 & 6+i \end{bmatrix}$

$\overline{AB} = \bar{A}\,\bar{B}$ $A \in \mathbb{C}^{m\times n}$ $B \in \mathbb{C}^{n\times p}$

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Complex Vectors

$u, v \in \mathbb{R}$
- lenght $\|v\| = \sqrt{v^T v}$ طول
- orthogonality $v \perp u$ $u^T v = 0$ تعامد
- angle $\cos(\theta) = \dfrac{u^T v}{\|u\|\,\|v\|}$

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for complex vectors

$$v = \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad v^T v = \begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = i^2 + 1 = -1 + 1 = 0$$

تعجب ! کرده

$$v^T v = \sum v_i^2$$

$$\|v\|^2 = \sum |v_i|^2 = \sum v_i \bar{v}_i$$

$$\left\| \begin{bmatrix} i \\ 1 \end{bmatrix} \right\|^2 = |i|^2 + |1| = 1 + 1 = 2 \quad \text{makes sense}$$

$v \in \mathbb{C}^n$

$$\|v\|^2 = \sum_{i=1}^{n} |v_i|^2 = \sum_{i=1}^{n} v_i \bar{v}_i = \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 & \text{—} & v_n \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ | \\ \bar{v}_n \end{bmatrix}$$

$$= \bar{v}^T v = v^T \bar{v}$$

$u, v \in \mathbb{R}^n$ $\langle u, v \rangle = u^T v = v^T u$ | $u, v \in \mathbb{C}^n$ $\langle u, v \rangle = \bar{v}^T u$

$$\langle v, u \rangle = \underbrace{\bar{u}^T v}_{1\times 1} = v^T \bar{u} = \overline{\bar{v}^T u} = \overline{\langle u, v \rangle} \quad \text{conjugate symmetry}$$

$u, v \in \mathbb{C}^n$ $\quad u \perp v \iff \bar{v}^T u = 0$

$u \in \mathbb{C}^n$ $\quad \|u\| = \sqrt{\langle u, u \rangle} = \sqrt{\bar{u}^T u}$

$\bar{u}^T = u^* = u^H$ Conjugate transpose

= Hermitian transpose

$U \in \mathbb{R}^{n\times n}$ orthogonal $U^TU = UU^T = I$

$U \in \mathbb{C}^{n\times n}$ $U = [u_1\ u_2 \cdots u_n] = \begin{bmatrix} u^{1T} \\ u^{2T} \\ \vdots \\ u^{nT} \end{bmatrix}$

$u_i \perp u_j$ $i \neq j$ $\langle u_i, u_j \rangle = \delta(i=j)$

$\|u_i\| = 1$ $1(i=j)$

$$\bar{u}_i^T u_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\begin{bmatrix} \bar{u}_1^T \\ \bar{u}_2^T \\ \vdots \\ \bar{u}_n^T \end{bmatrix} [u_1\ u_2 \cdots u_n] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$\bar{U}^T U = I$ Unitary Matrix

$\longrightarrow U\bar{U}^T = I$

$\bar{U}^T = U^* = U^H = U' \longrightarrow$ MATLAB

$U^*U = UU^* = I \Longrightarrow$ Unitary Matrix

A symmetric $\Longrightarrow A^T = A$

$A \in \mathbb{C}^{n\times n}$ Conjugate Symmetric $\bar{A}^T = A$

$A^* = A$

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 4 \end{bmatrix} \quad A^* = \begin{bmatrix} 2 & 1-i \\ 1+i & 4 \end{bmatrix}$$

$A \in \mathbb{R}^{n\times n}$ is symmetric $A^T = A$ LA24 V

Let (λ, v) be an eigenpair of A.
$\begin{pmatrix} \lambda \in \mathbb{C} \\ v \in \mathbb{C}^n \end{pmatrix}$

$$Av = \lambda v \Rightarrow \bar{v}^T A v = \lambda \bar{v}^T v = \lambda \|v\|^2 \quad \text{I}$$

$$\underbrace{\bar{v}^T A v}_{1\times 1} = (\underbrace{\bar{v}^T A v}_{1\times 1})^T = v^T A^T \bar{v} = v^T A \bar{v}$$

$$\overline{\bar{v}^T A v} = \overline{v^T A \bar{v}} = \bar{v}^T \bar{A} v \underset{A\in\mathbb{R}^{n\times n}}{=} \bar{v}^T A v \Rightarrow \boxed{\bar{v}^T A v \in \mathbb{R}} \quad \text{II}$$

$$\text{I} \Rightarrow \left.\begin{array}{l} \bar{v}^T A v = \lambda \|v\|^2 \\ \bar{v}^T A v \in \mathbb{R} \\ \|v\|^2 \end{array}\right\} \lambda \in \mathbb{R}$$

$\Rightarrow$ For any ~~symmetr~~ ~~real~~ real symmetric matrix A $(A \in \mathbb{R}^{n\times n}, A^T = A)$ all eigenvalues are real. (same thing is ~~ture~~ true for Hermitian matrices $A \in \mathbb{C}^{n\times n}$, $A^* = A$)

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Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix and (v_1, λ_1) & (v_2, λ_2) are eigenpairs with different eigenvalues, $\lambda_1 \neq \lambda_2$

$$A v_1 = \lambda_1 v_1 \Rightarrow \boxed{\bar{v}_2^T A v_1 = \lambda_1 \bar{v}_2^T v_1} \quad \text{I}$$

$$A v_2 = \lambda_2 v_2 \Rightarrow \bar{v}_1^T A v_2 = \lambda_2 \bar{v}_1^T v_2$$

$\lambda_1 \neq \lambda_2$

$$\Downarrow$$

$$\overline{\bar{v}_1^T A v_2} = \overline{\lambda_2 \bar{v}_1^T v_2}$$

$$\Downarrow$$

$$v_1^T A \bar{v}_2 = \bar{\lambda}_2 v_1^T \bar{v}_2$$

real

$$\bar{\lambda}_2 = \lambda_2 \Downarrow T$$

$$\boxed{\bar{v}_2^T A v_1 = \lambda_2 \bar{v}_2^T v_1} \quad \text{II}$$

$$\text{I}, \text{II} \Rightarrow \left.\begin{array}{l} \lambda_1 \bar{v}_2^T v_1 = \lambda_2 \bar{v}_2^T v_1 \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \bar{v}_2^T v_1 = 0 \Rightarrow v_1 \perp v_2$$

$A \in \mathbb{R}^{n\times n}$ Real symmetric $\Rightarrow$

Eigenvectors corresponding to different eigenvalues are orthogonal.