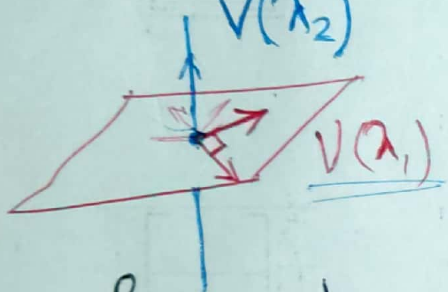
A real symmetric $A \in \mathbb{R}^{n\times n}$

$A^T = A$

$$\Rightarrow \begin{cases} \lambda_i \in \mathbb{R} \\ \lambda_i \neq \lambda_j \Rightarrow v_i \perp v_j \quad v_i^T v_j = 0 \end{cases}$$

$V_A(\lambda_i) \overset{def}{=}$ eigenspace corr. to λ_i

$\Rightarrow \lambda_i \neq \lambda_j \Rightarrow V_A(\lambda_i) \perp V_A(\lambda_j)$



$\forall v_2 \in V(\lambda_2)$: $A v_2 = \lambda_2 v_2$

$\forall v_1 \in V(\lambda_1)$: $A v_1 = \lambda_1 v_1$

for each eigenspace $V(\lambda_i)$ we can choose an orthonormal basis.

Diagonalizable Matrix $A \in \mathbb{R}^{n\times n}$

$\exists V \in \mathbb{R}^{n\times n}$ non-singular s.t. $A = V D V^{-1}$

diagonal

$V = [v_1 \; v_2 - v_n]$

Let $A \in \mathbb{R}^{n\times n}$ has n independent eigen~~values~~ vector
$\in \mathbb{C}^{n\times n}$

(we can choose n idependent eigenvectors ~~eigenvalues~~)

$\exists v_1, v_2 \cdots v_n \in \mathbb{C}^n$ s.t. $A v_i = \lambda_i v_i$, $\lambda_i \in \mathbb{C}$

& $v_1, v_2 \cdots v_n$ independent.

$$A[v_1, v_2, \cdots, v_n] = [Av_1, Av_2, \cdots, Av_n]$$

$$A[v_1, v_2, \dots, v_n] = [Av_1, Av_2, \dots, Av_n]$$

$V \in \mathbb{C}^{n\times n}$

$$= [\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_n v_n]$$

$$= \underbrace{[v_1, v_2, \dots, v_n]}_{V \in \mathbb{C}_{n\times n}} \underbrace{\begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ & & \ddots \\ \emptyset & & & \lambda_n \end{bmatrix}}_{\Lambda \in \mathbb{C}^{n\times n}}$$

$$\boxed{AV = V\Lambda}$$

$V = [v_1\ v_2 \cdots v_n] \in \mathbb{C}^{n\times n}$ the matrix of Eigenvectors

Λ the diagonal matrix of ~~Eigen~~ corresponding Eigenvalues

$v_1, v_2, \dots, v_n$ idependet $\Rightarrow V$ invertible

$\Rightarrow A = V\Lambda V^{-1} \Rightarrow A$ is diagonalizable.

Eigendecomposition of A

$$\boxed{V^{-1}AV = \Lambda}$$

diagonalize

A has an Eigendecomposition $\Longleftrightarrow$ A is diagonalizable.

A has $\underline{n}$ independent eigenvectors $\Longrightarrow A$ is diagonalizable

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A is diagonalizable $\Rightarrow A = VDV^{-1}$

$\Rightarrow$ ~~A?~~ $AV = VD$ & V non-singular (n indep. columns)

$$A[v_1\ v_2 - v_n] = [v_1\ v_2 - v_n]\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \Rightarrow Av_i = d_i v_i$$

$\Rightarrow n$ indep. eigenvectors

All real symmetric matrices are diagonalizable (n independent eigenvectors)

Same is true for Hermitian Matrices. $(A^* = A)$

Let $A \in \mathbb{R}^{n\times n}$ is symmetric.

(Let $A \in \mathbb{C}^{n\times n}$ is Hermitian (conjugate symmetric))

$\Rightarrow$ We can choose n orthonormal eigenvectors

$v_1, v_2, \dots, v_n$. $\quad V = [v_1\ v_2 \cdots v_n]$

if V real $\Rightarrow V^T V = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} [v_1\ v_2 \cdots v_n] = I \Rightarrow$

A is real symmetric

$V^T V = V V^T = I \Rightarrow V$ orthogonal

$V^{-1} = V^T$

$$A = V\Lambda V^{-1} = V\Lambda V^T$$

$x \in \mathbb{R}^n \Rightarrow x' = V^{-1}x = V^T x$ representation of x in the eigen basis $v_1, v_2, \dots v_n$.

$$x' = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} x = \begin{bmatrix} v_1^T x \\ v_2^T x \\ \vdots \\ v_n^T x \end{bmatrix} \qquad x'_i = v_i^T x$$

$$Ax = V\Lambda V^T x = V\Lambda \begin{bmatrix} v_1^T x \\ \vdots \\ v_n^T x \end{bmatrix} = \underset{[v_1 \cdots v_n]}{V} \begin{bmatrix} \lambda_1 v_1^T x \\ \lambda_2 v_2^T x \\ \vdots \\ \lambda_n v_n^T x \end{bmatrix} =$$

$$= v_1(\lambda_1 v_1^T x) + v_2(\lambda_2 v_2^T x) + \dots + v_n(\lambda_n v_n^T x)$$

$$= \lambda_1 (v_1 v_1^T) x + \lambda_2 v_2 v_2^T x + \dots + \lambda_n (v_n v_n^T) x.$$

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if $A \in \mathbb{C}^{n\times n}$ $A^* = A \Rightarrow V$ unitary $\Rightarrow V^* V = V V^* = I \Rightarrow A = V\Lambda V^*$

$\underline{\lambda_1, \lambda_2, \dots, \lambda_n}$ such than $(Av_i = \lambda_i v_i)$

such that $\lambda_i \neq \lambda_j \quad i \neq j$

$\Rightarrow$ n ~~dist~~ distinct eigenvalues.

Are $v_1, v_2, \dots, v_n$ independent?

$$\sum_{i=1}^{n} a_i v_i = 0 \Rightarrow \sum a_i A v_i = 0 \Rightarrow \sum a_i \lambda_i v_i = 0$$

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = 0$$

$$\Rightarrow A \sum a_i \lambda_i v_i = \sum a_i \lambda_i^2 v_i = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = 0$$

$A^T = A$

$A \in \mathbb{R}^{n \times n}$ $\Rightarrow$ $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ \emptyset & & \ddots \; \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$

$\lambda_1, \dots, \lambda_n$ real

Sometimes we like to have $\lambda_i \geq 0$

or

$\lambda_i > 0$

$A \in \mathbb{R}^{n \times n}$ is (symmetric) positive definite if for all $x \in \mathbb{R}^n$, $x \neq \vec{0}$ we have $A^T = A$ & $x^T A x > 0$.