$$\left.\begin{array}{l} A\in\mathbb{R}^{n\times n} \\ A^T=A \end{array}\right\} \quad A = V\Lambda V^T$$

Λ → diagonal, V → orthogonal $(V^TV=VV^T=I)$

$\Lambda, V \in \mathbb{R}^{n\times n}$

$$\left.\begin{array}{l} A\in\mathbb{C}^{n\times n} \\ A^*=A \end{array}\right\} \quad A = V\Lambda V^*$$

$\Lambda \in \mathbb{R}^{n\times n}$ diagonal

$V^*V = VV^* = I$

Positive definite مثبت معین $\boxed{A \succ 0}$

Symmetric $A^T=A$

$$A\in\mathbb{R}^{n\times n} \text{ positive definite} \begin{cases} A^T=A \\ \forall x\in\mathbb{R}^n,\ x\neq 0 \quad x^TAx>0 \end{cases}$$

Positive Semi-definite $\boxed{A \succeq 0}$

$A^T=A$

$\forall x\in\mathbb{R}^n \quad x^TAx\geq 0$

$$A\succ 0 \Longrightarrow A\succeq 0$$

Positive definite matrices $\subset$ positive semi-definite matrices

---

Negative definite $\quad A=A^T$

$\forall x\in\mathbb{R}^n,\ x\neq 0 \quad x^TAx<0$

Negativ semi-definite $\quad A=A^T$

$\forall x\in\mathbb{R}^n \quad x^TAx\leq 0$

For $A\in\mathbb{C}^{n\times n}$ positive-difinite $\begin{cases} A^*=A \\ \forall x\in\mathbb{C}^n,\ x\neq 0 \quad x^*Ax>0 \end{cases}$

for complex matrices

$A \succ 0$ (A is PD)

Can A be singular?

A singular $\Rightarrow \exists\, x \neq 0$, $x \in \mathbb{R}^n$ $Ax = 0 \Rightarrow x^T A x = 0$

$\Rightarrow$ A is not PD

تناقض!

All PD matrices are non-singular

---

for some $x \neq 0$ we have $x^T A x = 0$
& ($A = A^T$) A symmetric. Is A singular.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x^T A x = 0$$

A: positive-semidefinite, $x^T A x = 0$ for some $x \neq 0$

is A singular? $x^T V \Lambda V^T x = 0 \Rightarrow y \neq 0 \quad y^T \Lambda y = 0$

A symmetric $\Rightarrow A = V \Lambda V^T \Rightarrow A v_i = \lambda_i v_i$

A is PD $\Rightarrow$ what about it's eigenvalues?

Let λ be an eigenvalue of A. & $v \neq 0$ be
~~the~~ a corresponding eigenvector.

$$\left.\begin{array}{l} Av = \lambda v \Rightarrow v^T A v = \lambda v^T v \\ \text{A is PD} \Longrightarrow v^T A v > 0 \\ v \neq 0 \Longrightarrow v^T v = \|v\|^2 > 0 \end{array}\right\} \lambda > 0$$

A is PD $\Rightarrow$ All eigenvalues (> 0) are positive.

A is PSD $\Longrightarrow$ All eigenvalues (≥ 0) are nonnegative.

$$\left.\begin{array}{l} A \text{ real \& symmetric} \\ \text{all eigenvalues are positive} \end{array}\right\} \Longrightarrow A = V\Lambda V^T$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \quad \lambda_i > 0 \quad i = 1 - n$$

For any $x \neq \emptyset \in \mathbb{R}^n$ $\quad x^T A x = x^T V \Lambda V^T x$

$$= (V^T x)^T \Lambda (V^T x)$$

let $y = V^T x \Rightarrow y \neq 0$ $\quad x^T A x = y^T \Lambda y \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$x = Vy$

$x = 0$

$$y_1^2 \lambda_1 + y_2^2 \lambda_2 + \cdots + y_n^2 \lambda_n > 0$$

$\Rightarrow A$ positive definite

$\boxed{\text{A symmetric matrix } A \text{ is positive definite}} \Longleftrightarrow \boxed{\text{All eigenvalues are positive}}$

~~symmetric~~

$A = A^T$ is PSD $\Longleftrightarrow$ all eigenvalues are nonnegative.

$A \in \mathbb{R}^{m \times n}$ $\quad C \in \mathbb{R}^{n \times n}$ $\quad A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}$ $\quad a_i \in \mathbb{R}^n$

$$\boxed{C = A^T A}$$

$$= \begin{bmatrix} a_1 & a_2 & - & a_m \end{bmatrix} \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} = \sum_{i=1}^{m} a_i a_i^T$$

Choose an arbitrary $x \in \mathbb{R}^n$, $x \neq 0$

$$x^T C x = x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 \geq 0$$

C is PSD. (So is $C' = AA^T \in \mathbb{R}^{m \times n}$)

$$x^T C x = x^T \left(\sum a_i a_i^T\right) x = \sum x^T a_i a_i^T x = \sum_{i=1}^{m} (a_i^T x)^2 \geq 0$$

When is $A^T A$ positive definite?

$C = A^TA \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{m\times n}$, A has full column rank.

$$\Downarrow$$

~~rank~~

$\operatorname{rank}(A) = n$, $m \geq n$

$$\Downarrow$$

there are n independent rows

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \qquad A^T = \begin{bmatrix} a_1 & a_2 & \text{—} & a_m \end{bmatrix}$$

$C = A^TA$ A has full column rank $\Rightarrow \boxed{Ax \neq 0}$ for all $x \neq 0$

Choose any $x \in \mathbb{R}^n$, $x \neq 0 \Rightarrow$

$$x^TCx = x^TA^TAx \Rightarrow (Ax)^T(Ax) = \|Ax\| > 0$$

A^TA is positive definite.

Assume $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m \in \mathbb{R}^n$, $\operatorname{span}(a_1, \text{—}, a_m) = \mathbb{R}^n$

(there are n independent vector among $\vec{a}_1 \text{—} \vec{a}_m$)

Let $A^T = \begin{bmatrix} a_1 & a_2 & \text{—} & a_m \end{bmatrix}$.

$C = A^TA = \sum a_i a_i^T \in \mathbb{R}^{n\times n}$ is positive-definite

C is called the correlation matrix made of $a_1, a_2, \ldots, a_n$.

Let $\underline{a_1, a_2, \text{—}, a_m \in \mathbb{R}^n}$, $\vec{\mu} = \frac{1}{m}\sum_{i=1}^{m} a_i$,

$\bar{a}_i = a_i - \mu$

The correlation matrix of $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ is called the covariance matrix.

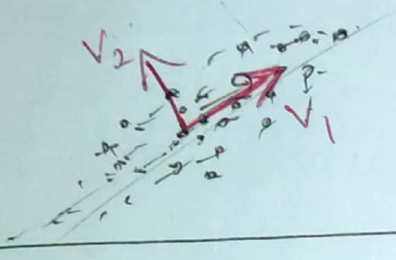
$a_1, a_2, \ldots, a_m$ $\quad \mu = \frac{1}{m}\sum_{i=1}^{m} a_i$

$\bar{a}_i = a_i - \mu$

$$\Sigma = \sum_{i=1}^{m} \bar{a}_i \bar{a}_i^T = \sum_{i=1}^{m} (a_i - \mu)(a_i - \mu)^T$$

covariance matrix is positive semidefinite

span$(\bar{a}_1, \ldots, \bar{a}_m) = \mathbb{R}^n \Rightarrow \Sigma$ is PD



Any positive semi-definite matrix P can be decomposed as $P = A^TA$. $\quad A \in \mathbb{R}^{m\times n}$

$P = A^T\underline{A} \qquad A \in \mathbb{R}^{n\times n}$

Let $H \in \mathbb{R}^{n\times n}$ is orthogonal

$$P = A^TA = A^T\underline{I}A = A^TH^{-1}HA = A^T\underline{H^TH}A$$
$$= (\underline{HA})^T(\underline{HA})$$
$$= A'^T A'$$

$P = A^TA \qquad A$: PSD

$= AA = A^2 \qquad A = \sqrt{P}$

$P = \underline{LL^T} \qquad L$ lower-triangular

Solving Linear Equation $\quad Ax = b \quad A \in \mathbb{R}^{n\times n}$ A is PD

$A = LL^T \qquad Ax = b = LL^Tx = b$

LA26

$$A = \begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{bmatrix}$$

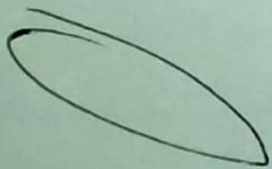
$$x^T A x = [x_1 \; x_2] \begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

VI

$$= x_1^2 + x_2^2 \pm 2\varepsilon x_1 x_2$$

$$x_1^2 + x_2^2 - 2x_1x_2 = (x_1 \mp x_2)^2$$

$$x_1^2 + x_2^2 + 2x_1x_2 = (x_1 + x_2)^2$$



$$\cancel{\leq x_1^2 + x_2^2 - 2\varepsilon x_1 x_2 \leq x_1^2 \pm x_2^2}$$