LA 27

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Eigenvalues $Av_i = \lambda_i v_i$ $A \in \mathbb{R}^{n\times n}$

$v_i \in \mathbb{R}^n$

Diagonalization

Eigenbasis $v_1, v_2, \dots, v_n$

$AV = V\Lambda$

$A = V\Lambda V^{-1}$ $\Lambda = V^{-1}AV$

$Ax = V\Lambda \underbrace{V^{-1}x}_{x'}$

$A \in \mathbb{R}^{n\times n} \rightarrow f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ $f(x) = Ax$

$f(x) = Ax$

$v_1, v_2, \dots, v_n$ $v_1, v_2, \dots, v_n$

only for square matrices
Eigenbasis might be complex
Eigenbasis might not be orthogonal
might not be diagonalizable

$A \in \mathbb{R}^{m\times n}$ $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$

$f(x) = Ax$ $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ $u_1, u_2, \dots, u_m \in \mathbb{R}^m$

$\boxed{y = Ax}$ $V = [v_1, v_2, \dots, v_n] \in \mathbb{R}^{n\times n}$ $U = [u_1, u_2, \dots, u_m] \in \mathbb{R}^{m\times m}$

x' = representation of x in basis $v_1, v_2, \dots, v_n = V^{-1}x$

y' = " " y in $u_1, u_2, \dots, u_m = U^{-1}y$

$\Rightarrow \left.\begin{array}{l} x = Vx' \\ y = Uy' \end{array}\right\}$ $Uy' = AVx' \Rightarrow y' = \underbrace{U^{-1}AV}_{\text{like it to be diagonal}} x'$

$y' = \underbrace{U^{-1}AV}_{m\times n} x'$

$y' \in \mathbb{R}^m$, $x' \in \mathbb{R}^n$

diagonal
$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 & 0 \end{bmatrix}$$

Like $v_1, v_2, \dots, v_n$ be orthonormal, so be $u_1, u_2, \dots, u_m$

$\Rightarrow U^{-1} = U^T, V^{-1} = V^T$

$y' = U^{-1}AVx' \Rightarrow y' = U^TAVx'$

We like $\left\{\begin{array}{l} U^TU = UU^T = I_{m\times m} \\ V^TV = VV^T = I_{n\times n} \\ \Sigma = U^TAV \text{ be diagonal} \end{array}\right.$

Eigen decomposition $A = V \underbrace{\Lambda}_{\text{diagonal}} \underbrace{V^{-1}}_{\text{invertible}}$

x-decomposition $A = U \Sigma V^{-1} = \underbrace{U}_{\text{orthogonal}} \underbrace{\Sigma}_{\text{diagonal}} \underbrace{V^T}_{\text{orthogonal}}$

$y = Ax \implies$ Change of basis $\vec{y} = x_1' \vec{u_1} + y_2' \vec{u_2} + \dots + y_m' \vec{u_n}$

$\vec{y} = U y' \Rightarrow y' = U^T y$

$x = V x' \Rightarrow x' = V^T x$

$y' = \underbrace{\Sigma}_{\text{diagonal}} x'$

$U^T y = \Sigma V^T x \xRightarrow[y = Ax]{} U^T A x = \Sigma V^T x$ for all $x \in \mathbb{R}^n$.

$\implies U^T A = \Sigma V^T$

$\implies AV = U\Sigma$

$y = Ax \Rightarrow U y' = AV x' \xRightarrow[y' = \Sigma x']{} U \Sigma x' = AV x'$ for all x'

$\implies U\Sigma = AV$

$$AV = U\Sigma \Rightarrow \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \dots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_l \end{bmatrix} \quad l = \min(m,n)$$

$i \leq \min(m,n) \implies A \underbrace{v_i}_{\text{left singular vector}} = \underbrace{\sigma_i}_{\text{singular value}} \underbrace{u_i}_{\text{right singular vector}}$

Eigenvalue $A v_i = \lambda_i v_i$ there are l $(1 \leq l \leq n)$ eigenvalues

Singular value $A v_i = \sigma_i u_i$ $\left.\begin{array}{l} u_1, \dots, u_m \\ v_1, \dots, v_m \end{array}\right\}$ ~~orthogonal~~ orthonormal

$$AV = U\Sigma \qquad A \in \mathbb{R}^{3\times 2}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

$Av_1 = \sigma_1 u_1$

$Av_2 = \sigma_2 u_2$

Eigenvalue $AV = V\Lambda \xrightarrow{\text{diagonalizable}} A = V\Lambda V^{-1}$

Singular Value $AV = U\Sigma \xrightarrow[\text{always}]{} A = U\Sigma V^T$

$A = U\Sigma V^T$: Singular Value Decomposition (SVD)

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)}$

U, V orthogonal

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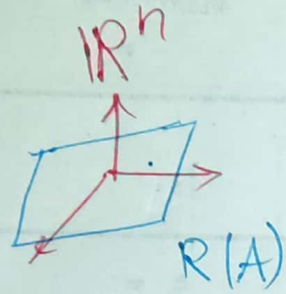
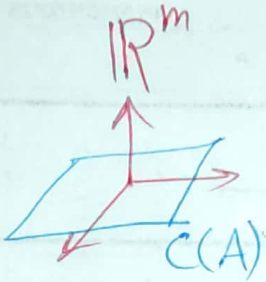
$A \in \mathbb{R}^{m \times n}$ $\quad$ $\text{rank}(A) = r$

$f(x) = Ax$ $\quad$ $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$



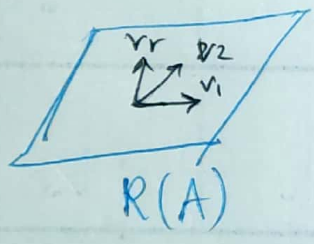


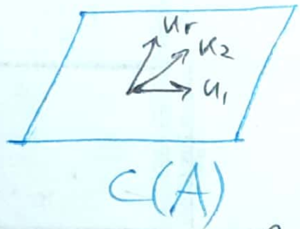
$\tilde{f}(x) = Ax$ $\quad$ $\tilde{f}: R(A) \longrightarrow C(A)$

$\forall x \in R(A) \quad \tilde{f}(x) = f(x)$

$\dim(R(A)) = \dim(C(A)) = r$

$\tilde{f}$ is one-to-one & onto (invertible)





$v_1, v_2, \ldots, v_r$ orthonormal basis for $R(A)$

$u_1, u_2, \ldots, u_r$ orthonormal " for $C(A)$

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