Singular Value Decomposition (SVD)

$A = U \Sigma V^T$

A: $m \times n$; U: orthogonal $m \times m$; Σ: diagonal $m \times n$; $V^T \longrightarrow$ orthogonal $n \times n$

$UU^T = U^TU = I_m$

$VV^T = V^TV = I_n$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_{\min(m,n)} \end{bmatrix}$$

$$\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \geqslant \cdots \geqslant \sigma_{\min(m,n)} \geqslant 0$$

$U = [u_1 \; u_2 \cdots u_m] \longrightarrow$ left singular vectors (منفرد)

$V = [v_1 \; v_2 \cdots v_n] \longrightarrow$ right singular vectors

$\sigma_1 \; \sigma_2 \cdots \sigma_{\min(m,n)}$ singular values

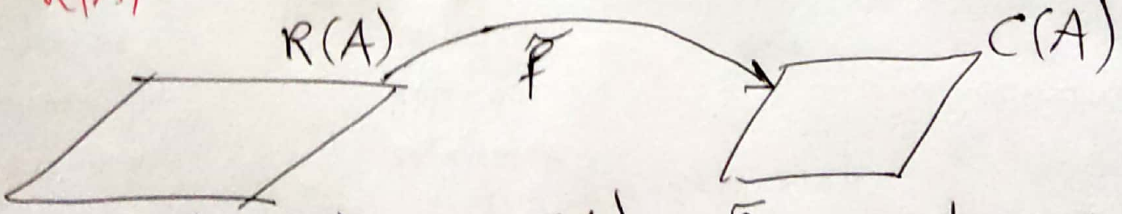
$$A v_i = \sigma_i u_i$$

$$A = [u_1 \; u_2 \cdots u_m] \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_l \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$= \sum_{i=1}^{\min(m,n)} \sigma_i u_i v_i^T = U \Sigma V^T$$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad f(x) = Ax$, A: $m \times n$

$R(A) \subseteq \mathbb{R}^n \qquad C(A) \subseteq \mathbb{R}^m$



$\tilde{f}: R(A) \longrightarrow C(A) \qquad \tilde{f}$ is one to one & onto

$\exists \tilde{f}^{-1}: C(A) \longrightarrow R(A)$

$\tilde{f}^{-1}(\tilde{f}(x)) = x \implies \tilde{f}^{-1}(f(x)) = x \quad \forall x \in R(A)$

$\tilde{f}(\tilde{f}^{-1}(y)) = f(\tilde{f}^{-1}(y)) = y \quad \forall y \in C(A)$

A $f(x) = Ax$

$$A = U\Sigma V^T = U\begin{bmatrix}\sigma_1 & & & 0 & 0\\ & \sigma_2 & 0 & 0 & 0\\ 0 & & \sigma_\ell & 0 & 0\end{bmatrix} V^T$$

$m \times n$

$$\Sigma^+ = \begin{bmatrix}1/\sigma_1 & & \emptyset\\ & 1/\sigma_2 & \\ \emptyset & & 1/\sigma_\ell\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$ → if not zero

$n \times m$

$$A^+ = V\Sigma^+U^T$$

$$A^+A = V\Sigma^+\underbrace{U^TU}\Sigma V^T = V\Sigma^+\Sigma V^T$$

$$= V\begin{bmatrix}\sigma_1^{-1} & & \emptyset\\ & \sigma_2^{-1} & \\ \emptyset & & 0\end{bmatrix}\begin{bmatrix}\sigma_1 & \emptyset & 0 & 0\\ \emptyset & \sigma_2 & 0 & 0\\ & & 0 & 0 & 0\end{bmatrix}V^T$$

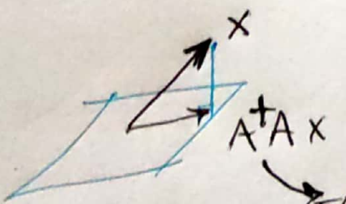
$$= V\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}V^T = \begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}\begin{bmatrix}v_1^T\\ v_2^T\\ v_3^T\end{bmatrix}$$

$$= \begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}1 & \\ & 1\end{bmatrix}\begin{bmatrix}v_1^T\\ v_2^T\end{bmatrix} = \begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}v_1^T\\ v_2^T\end{bmatrix}$$

$x \in R(A) \Rightarrow \alpha v_1 + \beta v_2$

$$x \in R(A) \Rightarrow A^+Ax = \begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}v_1^T\\ v_2^T\end{bmatrix}(\alpha v_1 + \beta v_2) = \begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$= \alpha v_1 + \beta v_2 = x$$



$A^T(AA^T)^{-1}A$

A full ~~column~~ row rank $\Rightarrow A^+ = A^T(AA^T)^{-1}$

$m > n$ $\quad \left[A\right] = \left[U\right]_{m\times m} \begin{bmatrix} \sigma_1 & & \emptyset \\ & \sigma_2 & \\ \emptyset & & \sigma_3 \\ & \emptyset & \end{bmatrix}_{m\times n} \left[V^T\right]_{n\times n}$

$\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \end{bmatrix}_{m\times n} \quad \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_m & \end{bmatrix}_{m\times n} \quad \sigma_1 \dots \sigma_{\min(m,n)}$

$$A = \underset{m\times m}{U} \underset{m\times n}{\Sigma} \underset{n\times n}{V^T}$$

$$AA^T = U\Sigma \underline{V^T V} \Sigma^T U^T$$

$$= U\Sigma\Sigma^T U^T$$

$\boxed{m=3}$ $= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & \boxed{\emptyset} \\ 0 & \sigma_2 & 0 & \\ 0 & 0 & \sigma_3 & \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix}$

$$AA^T = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_3^2 \end{bmatrix} \longleftrightarrow \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix}$$

$$\underbrace{AA^T}_{\text{symmetric \& positive semi def}} = \underbrace{U \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_3^2 \end{bmatrix} U^T}_{\text{eigen decomposition for symmetric matrices}} \quad U^T \rightarrow U^{-1}$$

$\Rightarrow (AA^T)u_i = \sigma_i^2 u_i$

$u_1, \dots, u_m$ eigenvectors of AA^T

$\sigma_1^2, \sigma_2^2, \dots, \sigma_m^2$ eigenvalues of AA^T

$AA^T =$

$v_1, v_2, \dots, v_n$ eigenvectors of A^TA

LA28 VI

Assume that we have a lot of data $x_1, x_2, \cdots, x_n \in \mathbb{R}^d$

$n = 1000000$

$d = 100$

arrange them in a matrix

$$X = \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{bmatrix} \in \mathbb{R}^{n \times d}$$

np.linalg.svd(A)

$$X = U \Sigma V^T \rightarrow \text{Full SVD}$$

$n \times n$ $n \times d$ $d \times d$

1000000×1000000 100×100

represented as

$$\vec{\sigma} = \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_d \end{bmatrix} \rightarrow 100$$

$\min(n,d) = d$

float32 $\Rightarrow 4 \times 10^{12}$ bytes

4000 GB

$$X = U \Sigma V^T = \begin{bmatrix} u_1 & u_2 & \cdots & u_d & u_{d+1} & \cdots & u_n \end{bmatrix} \begin{bmatrix} \ddots \\ 0 \end{bmatrix} [V^T]$$

$m \times n$

$m \gg n$

$$= \begin{bmatrix} u_1 & u_2 & \cdots & u_d \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{bmatrix} V^T$$

$4 \times 1000000 \times 100 = 400$ MB

$\ell = \min(m, n)$

$$A = U \Sigma V^T = U_{1:\ell} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_\ell \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_\ell^T \end{bmatrix}$$

skinny SVD, thin SVD