$A \quad \mathrm{rank}(A) = r < \min(m,n)$

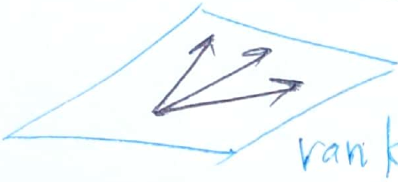
$$A = U \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0_0 \end{bmatrix} V^T \qquad A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{200 \times n}$$

In Practice we have $\tilde{A} = \underline{A} + \underline{N}$.

$$A = U \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0_0 \end{bmatrix} V^T$$

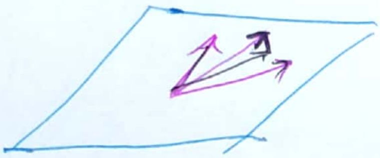
$$\tilde{A} = \tilde{U} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \tilde{V}^T$$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$



$\mathrm{rank}(A) = 2$

$$\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$$



$$\frac{\sigma_3}{\sigma_1}$$

$$10^5 \left\{ \overbrace{A}^{10^5} \right. = 10G$$

$$A = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_{100} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

Low-rank
Matrix Approximation

$$\simeq \underbrace{\begin{bmatrix} u_1 & \cdots & u_{100} \end{bmatrix}}_{100 \times 10^5} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_{100} \end{bmatrix}}_{100} \underbrace{\begin{bmatrix} v_1^T \\ \vdots \\ v_{100}^T \end{bmatrix}}_{100 \times 10^5}$$

$\simeq 20M$

$$\tilde{A} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \cong A' = \begin{bmatrix} \\ U \\ \\ \end{bmatrix} [\Sigma] [\quad V^T \quad]$$

$\tilde{A}$

find the closest matrix A to $\tilde{A}$
such that $\operatorname{rank}(A) \le r$.

$$\operatorname{dist}(A, \tilde{A})$$

$$\|A - \tilde{A}\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} - \tilde{a}_{ij})^2}$$

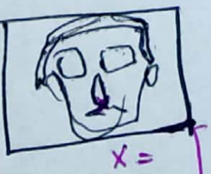
$$\|M\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij}^2} = \sqrt{\sum_{i=1}^{\min(m,n)} \sigma_i^2}$$

$$\|M\|_2 = \max \|Mx\|_2 \qquad \|x\|_2 = 1$$

$$= \sigma_{max} = \sigma_1$$

$$\tilde{A} = U \begin{bmatrix} \sigma_1 & & & \\ & \ddots \sigma_r & & \\ & & \sigma_{r+1} & \\ & & & \sigma \end{bmatrix} V^T$$

$$A = U \begin{bmatrix} \sigma_1 & & & \\ & \ddots \sigma_r & & \\ & & \emptyset & \\ & & & \emptyset \end{bmatrix} V^T$$



$x = \begin{bmatrix} m_1 \\ m_2 \\ 1 \end{bmatrix}$

$y = \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix}$

$x^T F y = 0$

F → Fundamental matrix.

$\operatorname{rank}(F) = 2$

$$f(x) = \frac{x^T A x}{x^T x} \qquad A \text{ symmetric}$$

$$x^*_{max} = \underset{x}{\arg\max} \frac{x^T A x}{x^T x}$$

$$= \underset{x}{\arg\max}\ x^T A x \quad \text{subject to} \quad x^T x = 1$$

$$x_{min} = \arg\min$$

$\cancel{x_{max} = \lambda_{max}(A)}$

$\cancel{\lambda_{max}}$ 24

A symmetric $\longrightarrow$ ~~eigen~~ n real eigenvector

Eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$

Corresponding Eigenvector $\vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad v_{n-1} \quad v_n$

$$x_{max} = v_1$$

$$x_{min} = v_n$$

$\|x\| = 1$

$$\vec{v}_1 = \underset{x}{\arg\max}\ x^T A x \quad \text{subject to} \quad x^T x = 1$$

$$\vec{v}_2 = \underset{x}{\arg\max}\ x^T A x \quad \text{s.t.} \quad x^T x = 1,\ x \perp v_1$$

$$\vec{v}_3 = \underset{x}{\arg\max}\ x^T A x \quad \text{s.t.} \quad x^T x = 1,\ x \perp v_1,\ x \perp v_2$$

$$A = V \Lambda V^T = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \cdots + \lambda_n v_n v_n^T$$

$$Ax = \lambda_1 v_1 v_1^T x + \lambda_2 v_2 v_2^T x + \cdots + \lambda_n v_n v_n^T x$$

$$\underset{m\times n}{y = Ax} \in \mathbb{R}^m \qquad x \in \mathbb{R}^n$$

$$x_{max} = \underset{x}{\operatorname{argmax}} \|Ax\|_2 \quad \text{subject to } \|x\|=1$$

$$= \underset{x}{\operatorname{argmax}} \|Ax\|_2^2 \quad \text{subject to } x^Tx=1$$

$$= \operatorname{argmax} (Ax)^T(Ax) \quad \text{" " "}$$

$$= \operatorname{argmax} x^T\underbrace{A^TA}x \quad \text{s.t. } x^Tx=1$$

x_{max} = the eigenvector of A^TA correspondi to the largest eigenvalue.

A^TA symmetric. & Positive Semidefinite

$\Longrightarrow x_{max} = v_1$ = the right singular vector of A correspond to largest singular value.

$A \in \mathbb{R}^{m\times n}$ $\quad x_{min} = v_n$

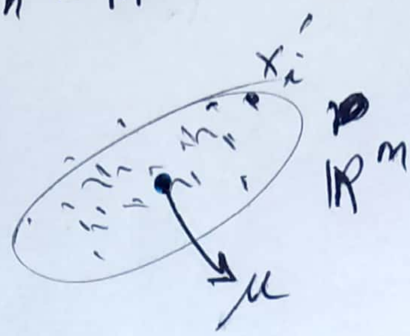
$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_{\min(m,n)} \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

x_{max} (v_1^T) $\quad x_{min}$ (v_n^T)

$$x = v_1 \Rightarrow \|Ax\| = \|\sigma_1 u_1\| = \sigma_1$$

$x_1', x_2', \dots, x_n' \in \mathbb{R}^m$

$X' = [x_1', x_2' \cdots x_n']$

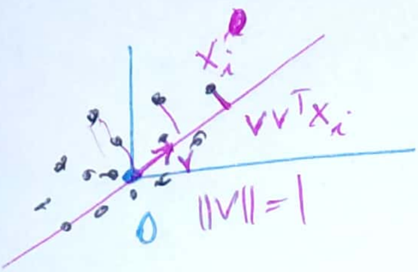


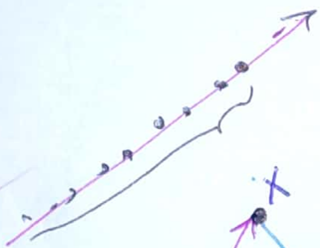
$$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i'$$

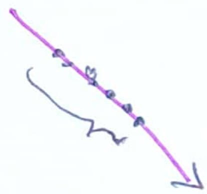
$$x_i = x_i' - \mu$$

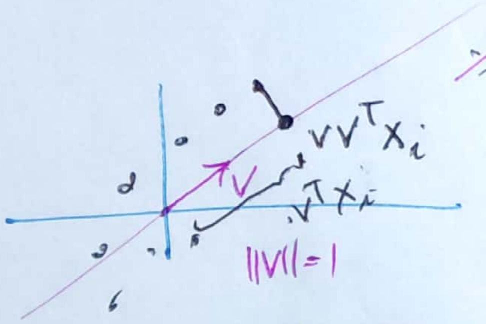
$x_1, x_2, \dots, x_n \in \mathbb{R}^m$

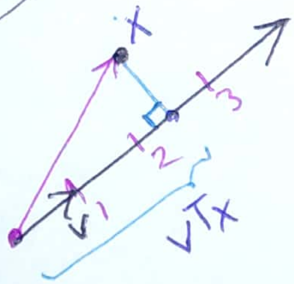
$$\frac{1}{n}\sum_{i=1}^{n} x_i = 0$$









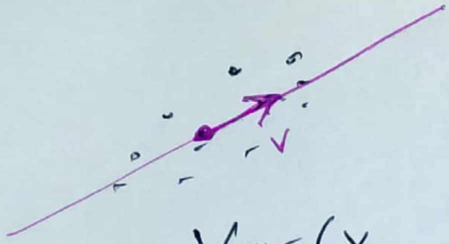


find the direction v such that the variance of the projected points is maximized

$$v = \arg\max_{v} \sum_{i=1}^{n} (v^Tx_i)^2 \quad \text{s.t. } \|v\|=1$$

$$\sum_{i=1}^{n} (v^Tx_i)^2 = \sum_{i=1}^{n} (v^Tx_i)(x_i^Tv)$$
$$= \sum_{i=1}^{n} v^T(x_ix_i^T)v$$
$$= v^T\Big(\underbrace{\sum_{i=1}^{n} x_ix_i^T}_{\text{covariance matrix}}\Big)v$$

LA29 (VI)



$Var(x_1 — x_n)$

$$var = v^T \underbrace{\Big(\sum \underset{n\times 1}{x_i}\, \underset{1\times n}{x_i^T}\Big)}_{n\times n} v$$

$\rightarrow$ covaricon

$$v^T \sum_{i=1}^{n} (x_i' - \mu)(x_i' - \mu)^T v$$

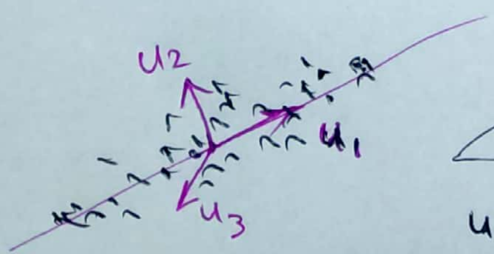
$$X = [x_1, x_2, \dots x_n]$$

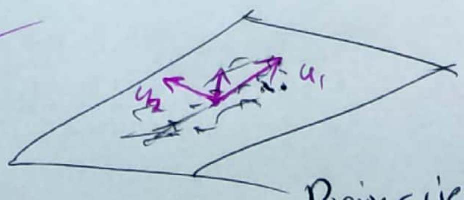
$$C = XX^T = [x_1 — x_n] \begin{bmatrix} x_1^T \\ | \\ x_n^T \end{bmatrix} = \sum_{i=1}^{n} x_i x_i^T$$

coraricnce matrix

$$v_{max} = \arg\max_{v} \; v^T (XX^T) v \quad s.t. \quad \|v\| = 1$$

v_{max} = the eigenvector of XX^T ~~covar~~
corresponding to the largest eigenvalue
= the first ~~lef~~ righ singular vector of X^T
= the first left singular vector of X $= u_1$





u_1, u_2, u_2 Principal Components

Principal Component Analysis (PCA)