$f: V \to U$

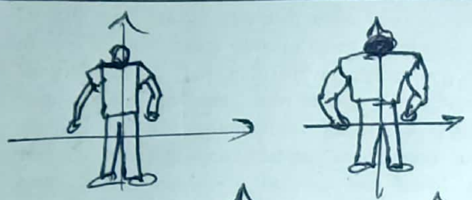
$$f(\alpha\vec{x}+\beta\vec{y}) = \alpha f(\vec{x})+\beta f(\vec{y})$$

$$f: \mathbb{R}^m \to \mathbb{R}^n$$

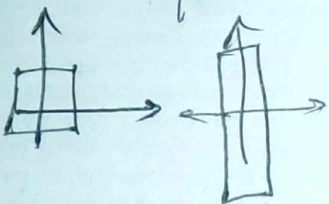
$$f(\vec{x}) = A\vec{x}$$

$$f(\vec{x}) = \alpha\vec{x} \qquad \begin{bmatrix}\alpha & 0\\ 0 & \alpha\end{bmatrix} = \alpha I_2$$

$$\begin{bmatrix}\alpha & & \emptyset\\ & \ddots & \\ \emptyset & & \alpha\end{bmatrix} = \alpha I_n$$



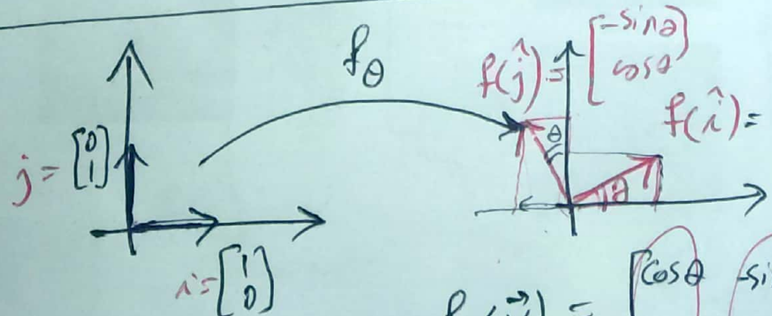
$$\begin{bmatrix}1.5 & \emptyset\\ \emptyset & 1\end{bmatrix}$$



$$\begin{bmatrix}\alpha & \emptyset\\ \emptyset & \beta\end{bmatrix}$$ diagonal matrix

$$\operatorname{diag}\left(\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix}\right) = \begin{bmatrix}\alpha_1 & & & \emptyset\\ & \alpha_2 & & \\ & & \ddots & \\ \emptyset & & & \alpha_n\end{bmatrix}$$

$$\operatorname{diag}(\vec{v}) = \begin{bmatrix}v_1 & & & \emptyset\\ & v_2 & & \\ & & \ddots & \\ \emptyset & & & v_n\end{bmatrix}$$

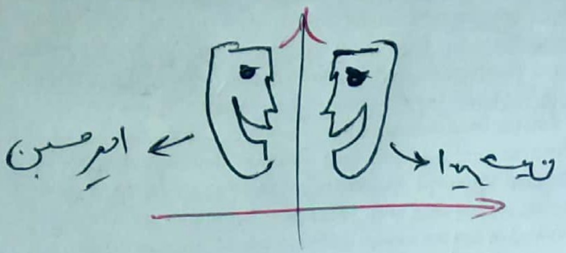


$$f(\hat{i}) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \qquad \vec{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

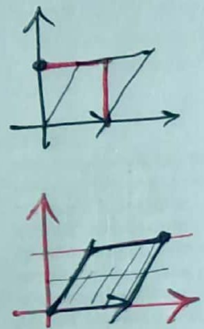
$$f_\theta(\vec{x}) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\vec{x}$$

$R^{-1} = R^T$; $R^T R = R R^T = I$ $\quad |R| = 1$

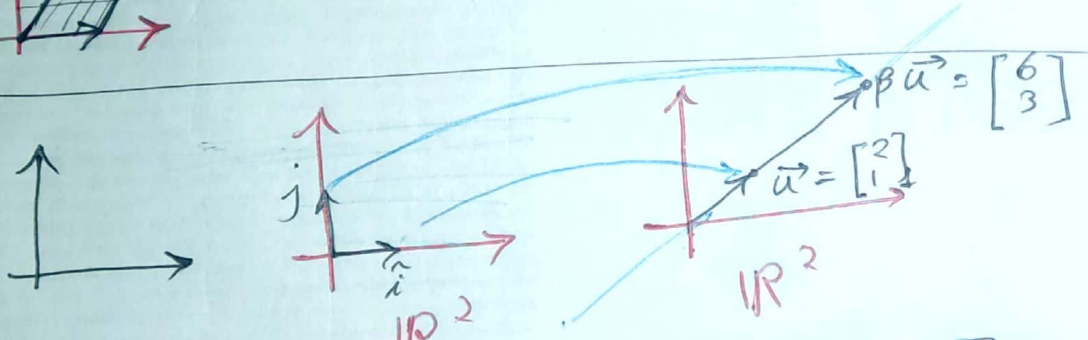
$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Shearing



$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+\alpha y \\ y \end{bmatrix}$$



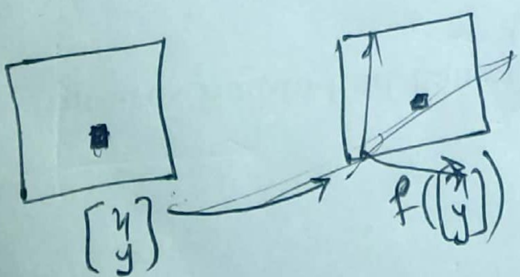
$$f(x) = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} x$$

$$f(\vec{x}) = \begin{bmatrix} \vec{u} & \beta\vec{u} \end{bmatrix} \vec{x}$$

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \vec{u} & \beta\vec{u} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x\vec{u} + y(\beta\vec{u})$$

$$(x+\beta y)\vec{u}$$



$f: V \to U$
$g: U \to W$

$h = g \circ f$

$h(x) = g(f(x)) = (g \circ f)(x)$

f, g linear $\to$ $g \circ f$ linear?

$y, x \in V$ $\quad g \circ f(x) = g(f(x))$

$$g \circ f(\alpha \vec{x} + \beta \vec{y}) = g(f(\alpha \vec{x} + \beta \vec{y}))$$
$$= g(\alpha f(x) + \beta f(y))$$
$$= \alpha g(f(x)) + \beta g(f(y))$$
$$= \alpha (g \circ f)(x) + \beta (g \circ f)(y)$$

$f(x) = Ax$
$g(y) = By$
$(g \circ f)(x) = g(f(x)) = g(Ax) = B(Ax) = (BA)x$

ضرب نقطه‌ای dot product $\quad x, y \in \mathbb{R}^m$

$m = 3$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\vec{x} \cdot \vec{y}$

inner product

$\langle \vec{x}, y \rangle$

$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$

$(\alpha x + \beta y) \cdot z$

$\langle \alpha \vec{x}, \vec{y} \rangle = \alpha \langle \vec{x}, \vec{y} \rangle$

$\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$

$\langle \alpha \vec{x} + \beta \vec{y}, \vec{z} \rangle = \alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle$

$\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

$\vec{x} \neq 0 \quad \langle \vec{x}, \vec{x} \rangle > 0$

$\langle \alpha \vec{x} + \beta \vec{y}, \vec{z} \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

$h(x, y) \quad h : V \times V \to \mathbb{R}$

$x, y \in V \quad h(x, y) = h(y, x)$ symetric

$x \neq 0 \Rightarrow h(x, x) > 0$ positive definite

$h(\alpha \vec{x} + \beta \vec{y}, \vec{z}) = \alpha h(\vec{x}, \vec{z}) + \beta h(\vec{y}, \vec{z})$ bilinear

Ex. $\mathbb{R} \to \mathbb{R}$ $\quad f(x, y) = 2xy$ → linear with respect to x (w.r.t.) → linear w.r.t. y

$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2xy$ → not linear

→ bilinear function دوخطی

$f\left(\alpha \begin{bmatrix} x \\ y \end{bmatrix}\right) \neq \alpha f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

$2(\alpha x)(\alpha y) \neq \alpha (2xy)$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$

$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$1 \times 3 \quad 3 \times 1$

$x \cdot y = x^T y = y^T x$

$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$

$= \vec{y}^T \vec{x}$