$A \in \mathbb{R}^{n \times n}$ $\quad$ $Ax = b$ $\quad$ $x \in \mathbb{R}^n$, $b \in \mathbb{R}^n$

$\text{rank}(A) = n$

$f(x) = Ax$ $\quad$ $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ $\quad$ is f linear?

$$f(x_1) = f(x_2) \Rightarrow Ax_1 = Ax_2 \Rightarrow Ax_1 - Ax_2 = \vec{0}$$

$$A\underbrace{(x_1 - x_2)}_{z} = 0 \Rightarrow Az = \vec{0} \Rightarrow \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \vec{0}$$

$$z_1\vec{a}_1 + z_2\vec{a}_2 + \cdots + z_n\vec{a}_n = 0 \Rightarrow z_1 = z_2 = \cdots = z_n = 0 \Rightarrow \vec{z} = \vec{0}$$

$$x_1 - x_2 = \vec{0} \Rightarrow x_1 = x_2 \quad \Rightarrow f \text{ is one-to-one}$$

---

is f onto?

Let $y \in \mathbb{R}^n$ ~~f(x) = y~~ is thera an $x \in \mathbb{R}^n$ such that $f(x) = y$?

$$f(x) = y \Leftrightarrow Ax = y = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = y$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = y$$

$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \in \mathbb{R}^n$ ~~form a basis for~~ span $\mathbb{R}^n$

$\Leftarrow \exists\, x_1, x_2, \ldots, x_n$ such that $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = y$

$\Rightarrow \exists\ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \exists x \quad Ax = y$

$$\left.\begin{array}{l} f: \mathbb{R}^n \rightarrow \mathbb{R}^n \quad f(x) = Ax \\ A \in \mathbb{R}^{n \times n} \text{ nonsingular} \end{array}\right\} \Rightarrow f \text{ is one to one \& onto}$$

$\Longrightarrow \exists\ f^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$ for all x

LA9 (II)

$f: U \to U$ is linear & invertible

$$\left(\exists f^{-1}\ \forall x\ \ f^{-1}(f(x)) = f(f^{-1}(x)) = x \right)$$

is f^{-1} linear?

$$f^{-1}(\alpha \vec{y}_1 + \beta \vec{y}_2) \overset{?}{=} \alpha f^{-1}(y_1) + \beta f^{-1}(y_2)$$

$$f(f^{-1}(\alpha y_1 + \beta y_2)) \overset{?}{=} f(\alpha f^{-1}(y_1) + \beta f^{-1}(y_2))$$

$$\alpha y_1 + \beta y_2 \overset{?}{=} \alpha f(f^{-1}(y_1)) + \beta f(f^{-1}(y_2))$$

$$\alpha y_1 + \beta y_2 = \alpha y_1 + \beta y_2$$

$$\left.\begin{array}{l} f(x) = \underline{A}x \\ \mathrm{rank}(A) = n \\ A \in \mathbb{R}^{n \times n} \\ f: \mathbb{R}^n \to \mathbb{R}^n \end{array}\right\} \quad \underbrace{f^{-1}}_{\mathbb{R}^n \to \mathbb{R}^n} \text{ is linear} \Rightarrow \exists B \in \mathbb{R}^{n \times n}$$

$$f^{-1}(y) = \underline{B}y \quad \forall y$$

$$\Rightarrow \quad f(f^{-1}(x)) = f^{-1}(f(x)) = x$$

$$A(Bx) = B(Ax) = x \quad \forall x$$

$$(AB)x = (BA)x = x \quad \forall x$$

$$\forall x \quad (\underline{AB})x = x \Rightarrow ABe_1 = e_1 \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbb{R}^3$$

$$ABe_2 = e_2 \qquad AB \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vdots$$

$$ABe_n = e_n$$

$$AB\,[e_1\ e_2 \cdots e_n] = [e_1\ e_2\ \ e_n]$$

$$ABI = I \Rightarrow AB = I, \qquad BA = I$$

$\forall A \in \mathbb{R}^{n \times n}$ non singalar $\exists B \quad AB = BA = I$

B is called the <u>inverse</u> of A
& is denoted by A^{-1}.

$Ax = b \quad A \in \mathbb{R}^{n \times n}$ <u>nonsingular</u>

$$A^{-1}Ax = A^{-1}b \Rightarrow \boxed{x = \underbrace{\underset{n\times n}{A^{-1}}\,\underset{n\times 1}{b}}_{n\times 1}}$$

there exist a <u>unique</u> <u>solution</u> $x = A^{-1}b$.

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rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \cancel{\cos\theta}\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

reflection

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scale

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A \in \mathbb{R}^{n\times n}$ singular

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null vector of A

$A \in \mathbb{R}^{m\times n}$ $\quad v \in \mathbb{R}^n$ $\quad Av = 0$

v is a null vector of A

$\vec{0} \in \mathbb{R}^n$ is a null vetor of any $A \in \mathbb{R}^{p\times n}$

Does $\underline{\underline{A}}$ have a $\underline{\text{nonzero}}$ / non-trivial null vector?

$$A \text{ non-singular} \Leftrightarrow A\underline{\underline{v}} = 0 \Rightarrow v = \vec{0}$$

A singular $\quad \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}$

$$\exists\, x \neq 0 \quad Ax = 0$$

$$\cancel{\vec{a}_i = z_1 a_1 + z_2 a_2 + \cdots + z_{i-1}\vec{a}_{i-1} + z_{i+1}\vec{a}_{i+1} + z_n \cdots}$$

$$\vec{a}_1 = \underline{z_2 \vec{a}_2 + \cdots + z_n \vec{a}_n}$$

$$0 = -1\,\vec{a}_1 + z_2 a_2 + \cdots + z_n \vec{a}_n$$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} -1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = 0$$

A singular $\quad \exists\, x \neq 0 \quad A\vec{x} = 0$

Assume A is $\underline{\text{singular}}$ & has an inverse A^{-1}.

$\quad \hookrightarrow \underline{\exists\, x \neq 0}, \ \underline{Ax = 0}$

$\underline{x} = Ix = A^{-1}\underline{Ax} = A^{-1}\underline{0} = 0$ تناقض