Linear Algebra for Computer Science
Homework 5

Read these first:

i You may write your solutions on paper, under a word processing software (MS-word, Libre Office, etc.), or under LaTeX.

ii If writing on paper, you must use a scanner device or a Camera Scanner (CamScanner) software to scan the document and submit a *single* PDF file.

iii Up to 15% extra score will be given to solutions written under LaTeX, provided that you follow either of the following conventions:

(a) Represent scalars with normal (italic) letters (a, A), vectors with bold lower-case letters ($\mathbf{a}$, using `\mathbf{a}`), and matrices with bold upper-case letters ($\mathbf{A}$, using `\mathbf{A}`), or

(b) represent scalars with normal (italic) letters (a, A), vectors with bold letters ($\mathbf{a}$, $\mathbf{A}$), and matrices with typewriter upper-case letters ($\mathtt{A}$, using `\mathtt{A}`).

(c) You latex document must contain a *title*, a *date*, and your name as the author.

(d) In all cases, you must submit a *single* PDF file.

(e) If writing under LaTeX, you must submit the *.tex* source (and other necessary source files if there are any) in addition to the PDF file.

Here is a short tutorial on LaTeX: `https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`

Questions

Eigenvalues and Eigenvectors

1. What is the relationship between the eigenvalues and eigenvectors of the square matrix $\mathtt{A}$ and those of $\mathtt{A} - \alpha\mathtt{I}$ where $\alpha \in \mathbb{R}$ and $\mathtt{I}$ is the identity matrix?

2. Prove that any eigenvalue of $\mathtt{A}$ is also an eigenvalue of $\mathtt{A}^T$. (Hint: use the characteristic polynomial).

3. The square matrix $\mathtt{A}$ is called (left) stochastic (or a Markov matrix) if its elements are nonnegative and its columns add up to 1 (programmatically sum(A,axis=0) == ones((1,n))). Prove that $\mathtt{A}$ has at least one unit eigenvalue $\lambda = 1$. (Hint: First prove that $\mathtt{A}^T$ has a unit eigenvalue.)

4. Let $\mathbf{v}$ be an eigenvector of $\mathtt{A}$ with a nonzero corresponding eigenvalue $\lambda \neq 0$. Prove that
   (a) $\mathbf{v}$ is in the column space of $\mathtt{A}$.
   (b) The (orthogonal) projection of $\mathbf{v}$ into the row space of $\mathtt{A}$ is nonzero. (Hint: decompose the vector as $\mathbf{v} = \mathbf{v}_r + \mathbf{v}_n$ where $\mathbf{v}_r$ and $\mathbf{v}_n$ are in the row space and null space of $\mathtt{A}$, respectively. Then show that $\mathbf{v}_r$ is nonzero)

5. Let $\mathtt{A}$ be a symmetric real matrix with real eigenvalues $1, 2, \ldots, n$, and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$. Prove that if $\lambda_i \neq \lambda_j$ then $v_i \perp v_j$.

Positive Definite Matrices

For all question in this section, by *positive definite* we mean *symmetric positive definite*.

6. Prove that a symmetric matrix is positive definite if and only if all its eigenvalues are positive. (Remember from the class that the eigen-decomposition of a symmetric matrix is in the form of $\mathtt{A} = \mathtt{V}\Lambda\mathtt{V}^{-1} = \mathtt{V}\Lambda\mathtt{V}^T$.)

7. Show that the diagonal elements of a positive definite matrix are all positive.

8. Remember that an operation $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined on a vector space $\mathcal{V}$ is an *inner product* if
   (a) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{u} \in \mathcal{V}$,
   (b) $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$,
   (c) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$,
   (d) $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$.

   Let $\mathtt{A} \in \mathbb{R}^{n \times n}$ be any *positive definite* matrix. Show that the operation $\langle \cdot, \cdot \rangle_{\mathtt{A}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

   $$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathtt{A}} = \mathbf{u}^T \mathtt{A} \mathbf{v}$$

   is indeed an inner product.