Linear Algebra for Computer Science
Homework 6

Read these first:

i You may write your solutions on paper, under a word processing software (MS-word, Libre Office, etc.), or under LaTeX.

ii If writing on paper, you must use a scanner device or a Camera Scanner (CamScanner) software to scan the document and submit a *single* PDF file.

iii Up to 15% extra score will be given to solutions written under LaTeX, provided that you follow either of the following conventions:

(a) Represent scalars with normal (italic) letters (a, A), vectors with bold lower-case letters ($\mathbf{a}$, using `\mathbf{a}`), and matrices with bold upper-case letters ($\mathbf{A}$, using `\mathbf{A}`), or

(b) represent scalars with normal (italic) letters (a, A), vectors with bold letters ($\mathbf{a}$, $\mathbf{A}$), and matrices with typewriter upper-case letters ($\mathtt{A}$, using `\mathtt{A}`).

(c) You latex document must contain a *title*, a *date*, and your name as the author.

(d) In all cases, you must submit a *single* PDF file.

(e) If writing under LaTeX, you must submit the *.tex* source (and other necessary source files if there are any) in addition to the PDF file.

Here is a short tutorial on LaTeX: `https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`

Questions

Singular Value Decomposition

1. Let $\mathtt{A}$ be a nonsingular square matrix and $A = \mathtt{U}\Sigma\mathtt{V}^T$ be its (full) SVD. Prove that $\det(\mathtt{U})\det(\mathtt{V}) = \text{sign}(\det(\mathtt{A}))$, that is $\det(\mathtt{U})\det(\mathtt{V}) = 1$ if $det(\mathtt{A}) > 0$ and $\det(\mathtt{U})\det(\mathtt{V}) = 1$ if $\det(\mathtt{A}) < 0$.

2. Show that for a symmetric positive definite matrix the eigenvalue decomposition $\mathtt{A} = \mathtt{V}\Lambda\mathtt{V}^{-1} = \mathtt{V}\Lambda\mathtt{V}^T$ is the same as its singular value decomposition.

3. Find a way to obtain the SVD of a symmetric matrix from its eigenvalue decomposition $\mathtt{A} = \mathtt{V}\Lambda\mathtt{V}^T$. Notice that the diagonal elements of Λ might be negative.

4. Consider a matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$ and two orthogonal matrices $\mathtt{P} \in \mathbb{R}^{m\times m}$ and $\mathtt{Q} \in \mathbb{R}^{n\times n}$. Show that the singular values of $\mathtt{PAQ}$ is the same as the singular values of $\mathtt{A}$.

Multivariate Calculus

5. Show that for a matrix $\mathtt{A} \in \mathbb{R}^{n\times n}$ the gradient of the expression $\mathbf{x}^T\mathtt{A}\mathbf{x}$ is equal to $(\mathtt{A} + \mathtt{A}^T)\,\mathbf{x}$. What is the gradient when $\mathtt{A}$ is symmetric?

6. Show that for a symmetric matrix $\mathtt{B}$ the gradient of $1/(\mathbf{x}^T\mathtt{B}\mathbf{x})$ with respect to $\mathbf{x}$ is $-2\mathtt{B}\mathbf{x}/(\mathbf{x}^T\mathtt{B}\mathbf{x})^2$ (if the gradient exists at $\mathbf{x}$).

7. Show that for symmetric matrices $\mathtt{A}$ and $\mathtt{B}$ the gradient of $f(\mathbf{x}) = (\mathbf{x}^T\mathtt{A}\mathbf{x})/(\mathbf{x}^T\mathtt{B}\mathbf{x})$ with respect to x is equal to

$$2(\mathtt{A}\mathbf{x}(\mathbf{x}^T\mathtt{B}\mathbf{x}) - \mathtt{B}\mathbf{x}(\mathbf{x}^T\mathtt{A}\mathbf{x}))/(\mathbf{x}^T\mathtt{B}\mathbf{x})2 = 2(\mathtt{A}\mathbf{x} - f(\mathbf{x})\,\mathtt{B}\mathbf{x})/(\mathbf{x}^T\mathtt{B}\mathbf{x}),$$

   if the gradient exists at $\mathbf{x}$.

8. Let $\mathtt{A}$ be symmetric. Calculate the gradient of $\exp(-\mathbf{x}^T\mathtt{A}\mathbf{x})$ with respect to $\mathbf{x}$.

9. Let $\mathtt{A}$ be (symmetric) positive definite. Compute the gradient of $\log(1 + \mathbf{x}^T\mathtt{A}\mathbf{x})$ with respect to $\mathbf{x}$.

10. Consider the function $f(\mathbf{x}) = \mathbf{x}^T\mathtt{A}\mathbf{x}/\left\|\mathbf{x}\right\|^2 = \mathbf{x}^T\mathtt{A}\mathbf{x}/(\mathbf{x}^T\mathbf{x})$ defined for a symmetric matrix $\mathtt{A}$. Show that the critical points of f are exactly the eigenvectors of $\mathtt{A}$. The critical points of a function f are points $\mathbf{x}$ at which the gradient is zero or nonexistant.

11. Consider the function $f(\mathbf{x}) = \mathbf{x}^T\mathtt{A}\mathbf{x}/(\mathbf{x}^T\mathtt{B}\mathbf{x})$ defined for symmetric matrices $\mathtt{A}$ and $\mathtt{B}$. Show that if $\mathtt{B}$ is invertible then the critical points of f are either the points for which $\mathbf{x}^T\mathtt{B}\mathbf{x} = 0$ or the eigenvectors of $\mathtt{B}^{-1}\mathtt{A}$.