Linear Algebra for Computer Science

Lecture 3

Span, Independence, Basis, Coordinates

Linear combination

Let $a,b \in R$. The vector $a\,x + b\,y$ is a linear combination of the vectors x and y.

Let $a_i \in R$. The vector $a_1 x_1 + a_2 x_2 + ... + a_n x_n$ is a linear combination of the vectors $x_1, x_2, ..., x_n$.

Span

$\text{span}(x,y) = \{ a\,x + b\,y \mid a,b \in R\}$

The space of all linear combinations of x and y.

$\text{span}(x_1, x_2, \ldots, x_n) = \{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \mid a_i \in R \}$

Span

We say that $x_1, x_2, \ldots, x_n$ span S if $S = \mathrm{span}(x_1, x_2, \ldots, x_n)$.

Linear dependence

K. N. Toosi
University of Technology

x,y,z are dependent if

- $x \in \text{span}(y,z)$, OR
- $y \in \text{span}(z,x)$, OR
- $z \in \text{span}(x,y)$

that is

- $x = a\,y + b\,z$, for some a, b, OR
- $y = a\,z + b\,x$, for some a,b, OR
- $z = a\,x + b\,y$, for some a,b.

Linear dependence

$x_1, x_2, \ldots, x_n \in V$ are **linearly dependent** if one of them can be written as a linear combination of the others (one of them is in the span of the others).

Linear independence

K. N. Toosi
University of Technology

x, y, z are independent if

- $x \notin \text{span}(y,z)$, AND
- $y \notin \text{span}(z,x)$, AND
- $z \notin \text{span}(x,y)$

Linear independence

$x_1, x_2, \ldots, x_n \in V$ are **linearly independent** if none of them can be written as a linear combination of the others.

Linear independence

K. N. Toosi
University of Technology

$x_1, x_2, \dots, x_n \in V$ are **linearly independent** if none of them can be written as a linear combination of the others.

Equivalently:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0 \Rightarrow a_1 = a_2 = \dots = a_n = 0$$

Basis

$v_1, v_2, \ldots, v_n \in V$ such that

- $v_1, v_2, \ldots, v_n$ are linearly independent
- $v_1, v_2, \ldots, v_n$ span V

Basis

$v_1, v_2, \ldots, v_n \in V$ such that

- $v_1, v_2, \ldots, v_n$ are linearly independent
- $v_1, v_2, \ldots, v_n$ span V

* n is the same for any choice of the basis vectors

* n is called the dimension of V

* There are also **infinite dimensional** vector spaces

* Basis (general definition)

$\{v_i\}_{i\in I} \subseteq V$ such that

- v_i's are linearly independent
- for any $v \in V$ there is a **finite** set of vectors $v_1, v_2, \ldots, v_d \in \{v_i\}_{i\in I}$ such that $v \in \mathbf{span}(v_1, v_2, \ldots, v_d)$

* Any vector space has a basis

* cardinality of $\{v_i\}_{i\in I}$ is the same for any choice of the basis vectors

* cardinality of $\{v_i\}_{i\in I}$ is called the dimension of V

Bases and Coordinate Representation

Why is independence needed? => uniqueness

every $x \in V$ can be written **uniquely** as a linear combination of the basis vectors $v_1, v_2, \ldots, v_n$.

Bases and Coordinate Representation

K. N. Toosi
University of Technology

⟹ Every $x \in V$ can be written as a <u>unique</u> linear combination of $u_1, \ldots, u_n$.

$$x = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$$

⟹ x can be represented as $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$.

$$x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$ as an array of real numbers.

a_i-s are called coordinates of x

مختصات

مختصات به بردارهای پایه وابسته است

Example: The Euclidean space

K. N. Toosi
University of Technology