Linear Algebra for Computer Science

Lecture 4

Matrices, Linear Subspaces, Column Space, Row Space

Matrices

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$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 7 & 2 & -1 & 3 \\ 1.5 & 7 & 4 & 5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a[0,0] & a[0,1] & a[0,2] & a[0,3] \\ a[1,0] & a[1,1] & a[1,2] & a[1,3] \\ a[2,0] & a[2,1] & a[2,2] & a[2,3] \end{bmatrix}$$

$$\in \mathbb{R}^{3\times 4} \qquad \in \mathbb{Z}^{m\times n} \qquad \in \mathbb{C}^{m\times n}$$

Vectors as special matrices?

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$$v = \begin{pmatrix} 1 \\ 0.5 \\ 7 \\ 2 \end{pmatrix} \in \mathbb{R}^4 \quad \in \mathbb{R}^n \quad \in \mathbb{C}^n$$

column vector $\begin{bmatrix} 1 \\ 0.5 \\ 7 \\ 2 \end{bmatrix} \in \mathbb{R}^{n \times 1}$

row vector $\begin{bmatrix} 1 & 0.5 & 7 & 2 \end{bmatrix} \in \mathbb{R}^{1 \times n}$

Multiplying a matrix by a vector

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$$\begin{bmatrix} 1 & 2 & 7 & 4 \\ -1 & 0 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ -10 \\ 30 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 60 \end{bmatrix} = \begin{bmatrix} 1 \times 10 + 2 \times 20 + 7 \times (-10) + 4 \times 30 \\ (-1) \times 10 + 0 \times 20 + 2 \times (-10) + 1 \times 30 \\ 3 \times 10 + 2 \times 20 + 1 \times (-10) + 0 \times 30 \end{bmatrix}$$

$A \in \mathbb{R}^{3 \times 4}$

$u \in \mathbb{R}^4$ $\quad v \in \mathbb{R}^3$

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \times (10) + \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \times (20) + \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix} \times (-10) + \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \times (30)$$

linear Combination of columns of A

Multiplying a matrix by a vector

linear combination of columns of A

$$\begin{array}{cc} & 3 \\ m & \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \\ & m \times 3 \end{array} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3 \in \mathbb{R}^m$$

$\mathbb{R}^{m \times 1}$

$a_i \in \mathbb{R}^m$

Multiplying a matrix by a vector

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linear combination of columns of A

$$\begin{array}{cc} & 3 \\ m & \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \\ & m\times 3 \end{array} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3 \in \mathbb{R}^m$$

$\mathbb{R}^{m\times 1}$

$a_i \in \mathbb{R}^m$

Define span in matrix form

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$$\operatorname{span}(u_1, u_2, \dots, u_n) = \{ \underbrace{\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n}_{\sum_{i=1}^{n} \alpha_i u_i} \mid \alpha_i \in \mathbb{R} \}$$

$$u_i \in \mathbb{R}^m$$

$$= \left\{ \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \;\middle|\; \alpha_i \in \mathbb{R} \right\} = \{ Ua \mid a \in \mathbb{R}^n \}$$

$$U \in \mathbb{R}^{m \times n} \qquad a \in \mathbb{R}^n$$

where $U = [u_1, u_2, \dots, u_n]$

$$U = [u_1 \; u_2 \; \cdots \; u_n] \in \mathbb{R}^{m \times n}$$

Column space

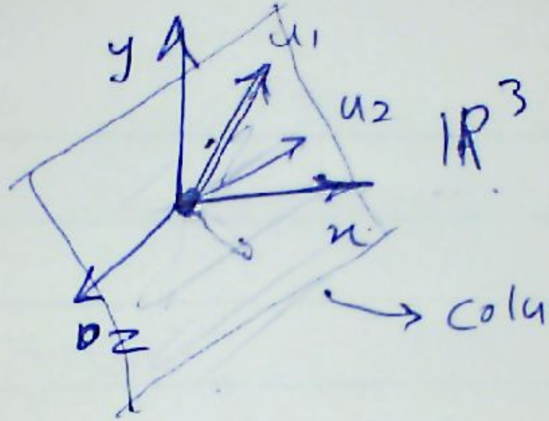
$$\underline{\underline{U}} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$$\begin{aligned}
&\operatorname{span}(u_1, u_2, \ldots, u_n) \\
&= \text{column space of } U \\
&= \text{range of } U \\
&= \{ Ua \mid a \in \mathbb{R}^n \}
\end{aligned}$$

Column space

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

$$u_1, u_2 \in \mathbb{R}^3$$



→ column space of U

Linear Subspace

Linear Subspace

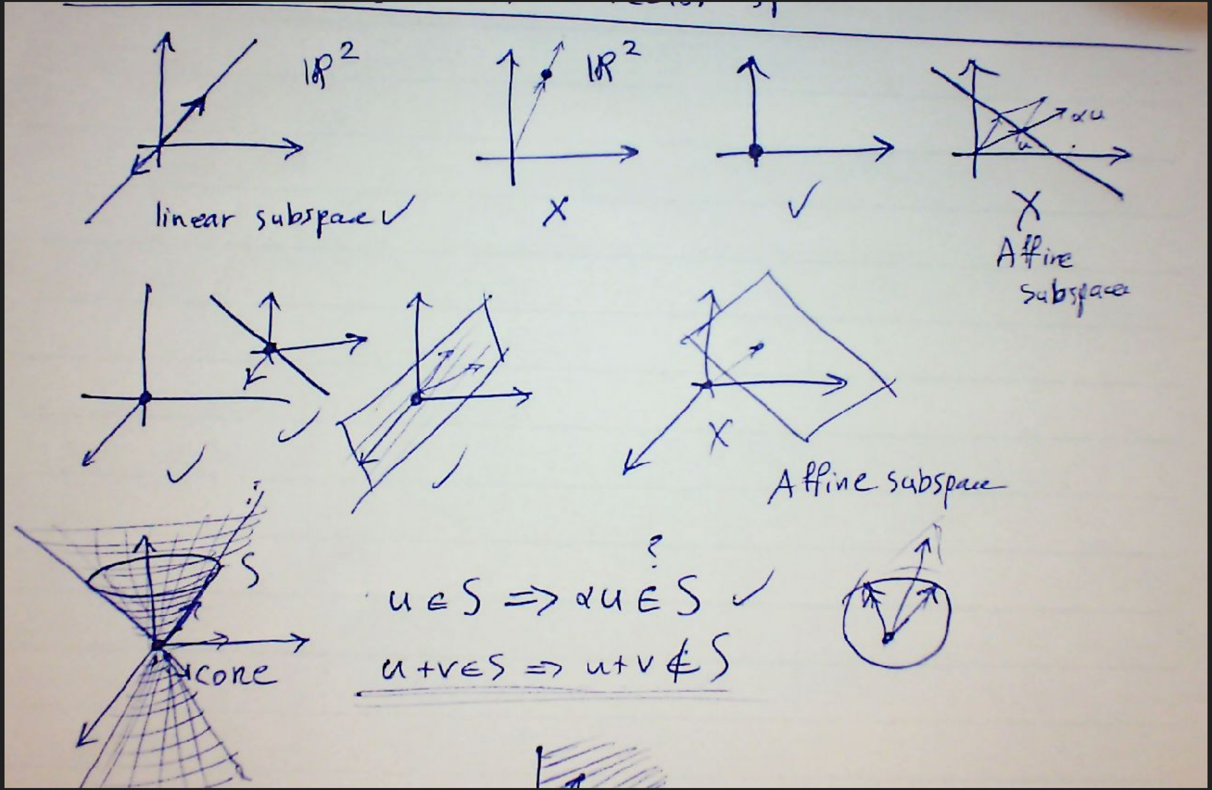
Let V be a vector space. A subset $S \subseteq V$ is called a linear subspace of V if

- For all $\mathbf{u} \in S$ and all $\alpha \in R \Rightarrow \alpha\,\mathbf{u} \in S$.
- For all $\mathbf{u},\mathbf{v} \in S \Rightarrow \mathbf{u} + \mathbf{v} \in S$.

Alternatively,

- For all $\mathbf{u},\mathbf{v} \in S$ and $\alpha, \beta \in R \Rightarrow \alpha\,\mathbf{u} + \beta\,\mathbf{v} \in S$.

Linear Subspace



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Data as columns of the matrix row-major programming languages

$u, v \in S \quad u+v \in S$ ✓

$\forall \alpha \geq 0 \quad \alpha u \in S$
$u \in S$ ×

scikit learn

$u_1, u_2, \dots, u_n \qquad U = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \qquad U[:,0] \quad U[:,1]$

python row major

C/C++

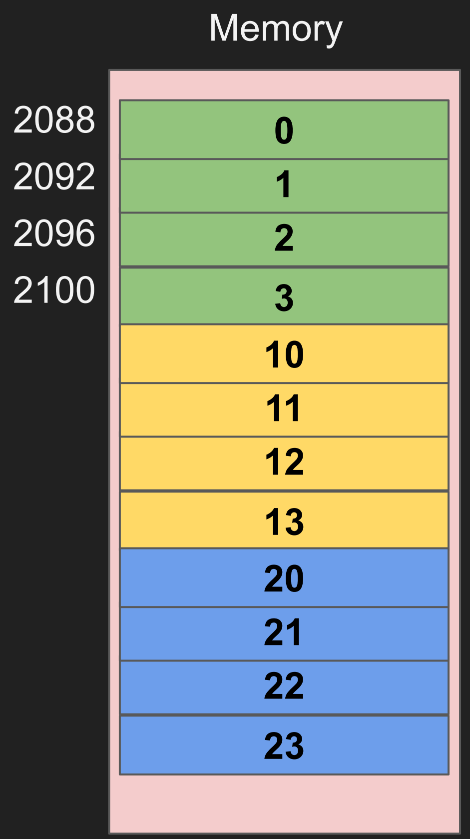
$U =$

$$U = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \\ a_3 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad u_1, u_2, \dots, u_n \in \mathbb{R}^m \qquad \mathbb{R}^{m \times 1}$

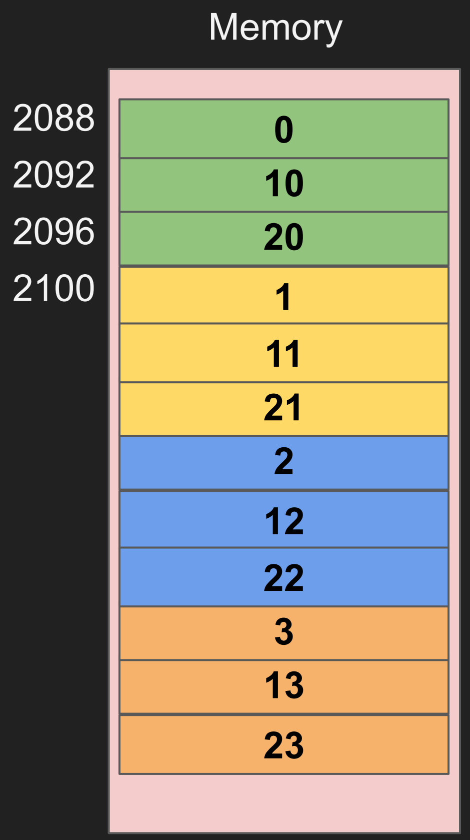
$\begin{bmatrix} u^T \end{bmatrix}$

Representing Data and Data Batches



row by row

	0	1	2	3
0	0	1	2	3
1	10	11	12	13
2	20	21	22	23



column by column

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Memory

2088	0
2092	1
2096	2
2100	3
	10
	11
	12
	13
	20
	21
	22
	23

Row Major

	0	1	2	3
0	0	1	2	3
1	10	11	12	13
2	20	21	22	23

Memory

2088	0
2092	10
2096	20
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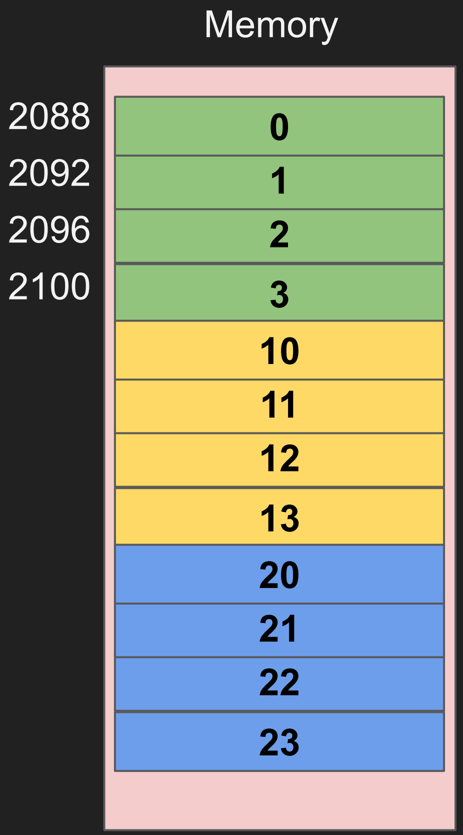
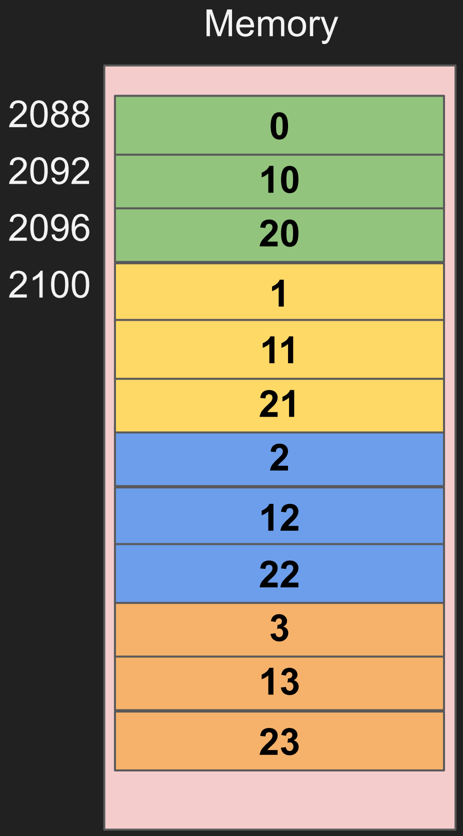
	23

Column Major



Row Major
(C,C++,Pascal, Python-numpy)

	0	1	2	3
0	0	1	2	3
1	10	11	12	13
2	20	21	22	23



Column Major
(Fortran, Matlab, R, …)

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Data as rows of the matrix
row-major programming languages

d_2

a_3

$$U = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

$$U \in \mathbb{R}^{n \times m}$$

n = U.shape[0] : # of data (feature vectors)

m = U.shape[1] : # of features

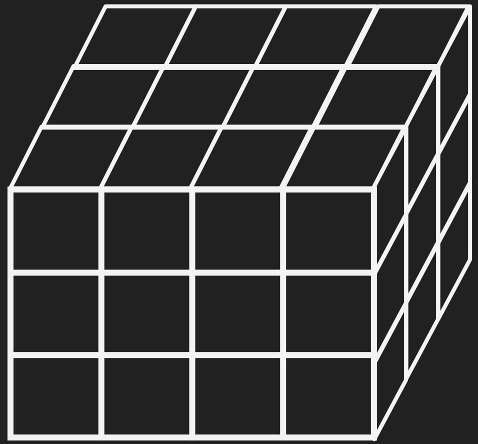
U[i,:]   U[i]

$u_i \in \mathbb{R}^m$ $u_i^T \in \mathbb{R}^{1 \times m}$

ND Arrays: Row-major vs. Column-major

- What does row-major and column-major mean?
    - Matlab vs Numpy ND-arrays

- **Row-major: last index moves fastest**



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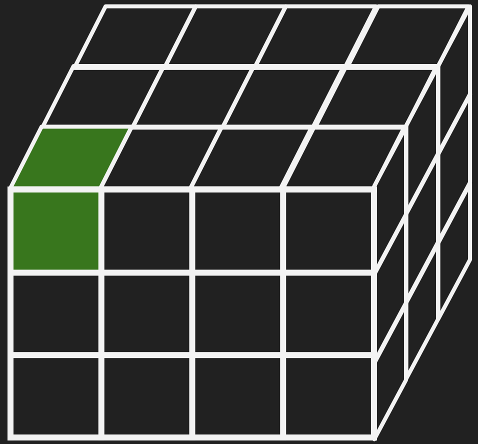

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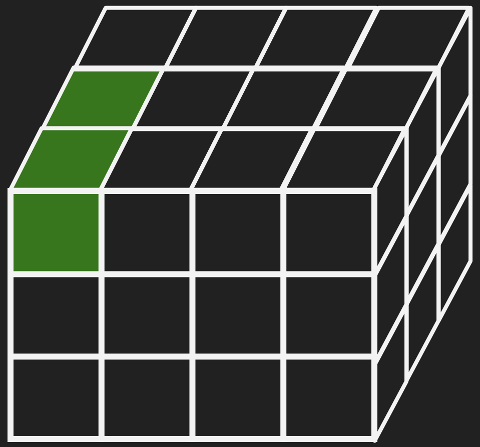

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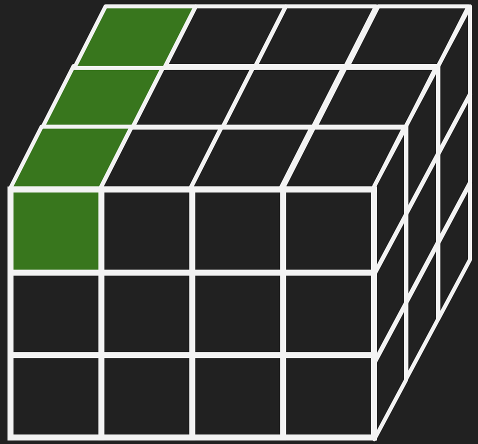

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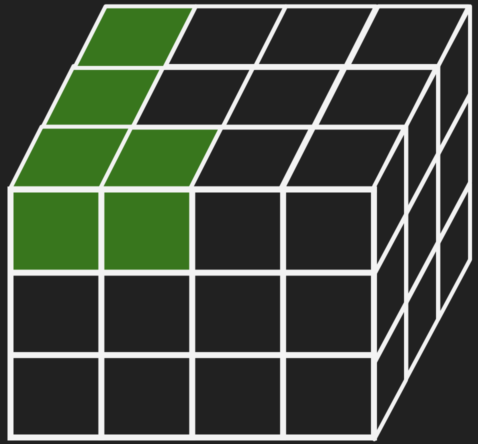

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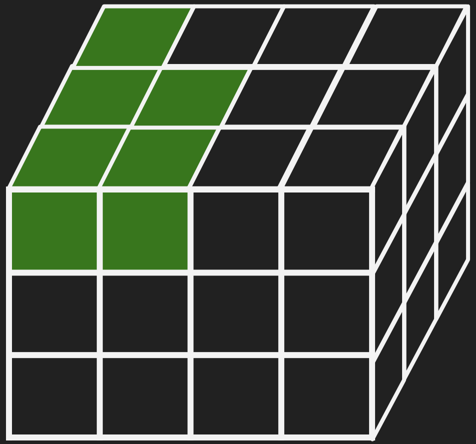

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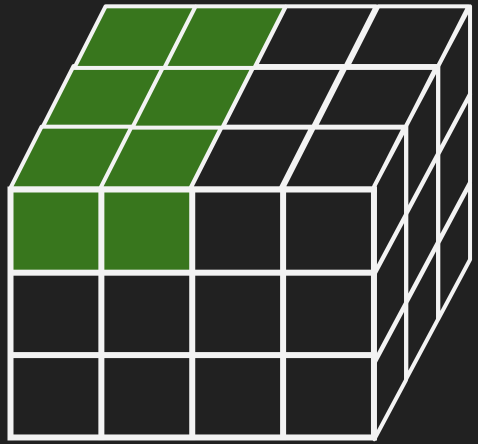

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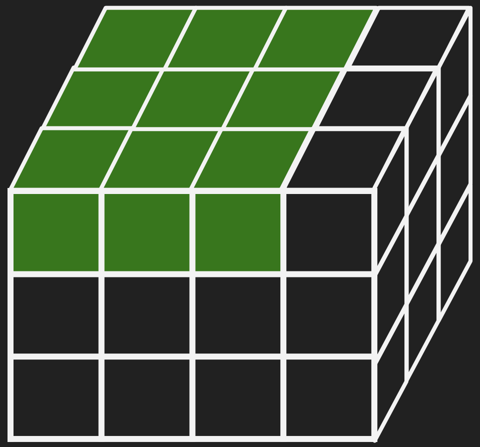

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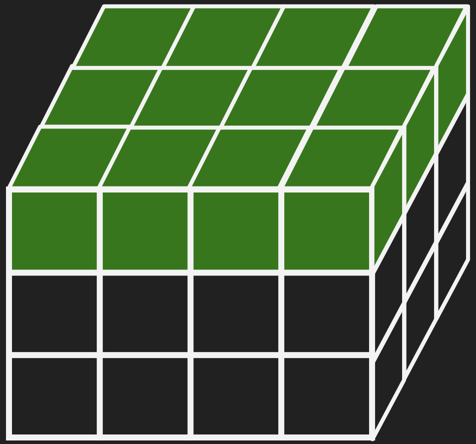

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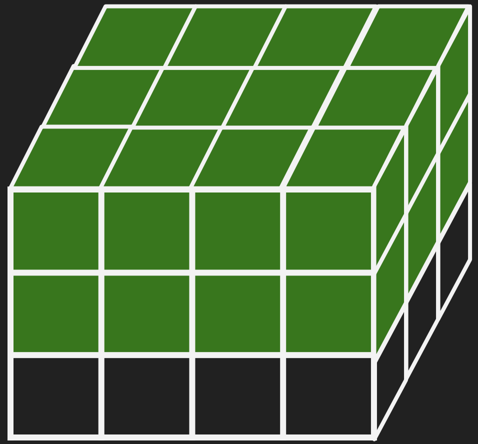

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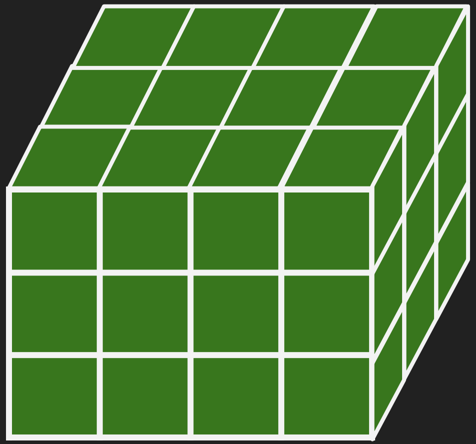

- **Row-major: last index moves fastest**

Linear combination of matrix rows

$U[i,:] \quad U[i]$

$u_i \in \mathbb{R}^m \quad u_i^T \in \mathbb{R}^{1\times m}$

$$\underset{\mathbb{R}^{1\times n}}{\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}} \underset{\mathbb{R}^{n\times m}}{\begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}} = \underbrace{\alpha_1 u_1^T + \alpha_2 u_2^T + \cdots + \alpha_n u_n^T}_{\in \mathbb{R}^{1\times m}} \in \mathbb{R}^{1\times m}$$

Span of the rows of the matrix

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$$\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{bmatrix}\begin{bmatrix}u_1 & u_2\\ v_1 & v_2\\ w_1 & w_2\end{bmatrix} = \begin{bmatrix}\alpha_1 u_1+\alpha_2 v_1+\alpha_3 w_1 & \alpha_1 u_2+\alpha_2 v_2+\alpha_3 w_2\end{bmatrix}$$

$$\alpha_1\begin{bmatrix}u_1 & u_2\end{bmatrix}+\alpha_2\begin{bmatrix}v_1 & v_2\end{bmatrix}+\alpha_3\begin{bmatrix}w_1 & w_2\end{bmatrix}$$

$$\alpha_1 u^T+\alpha_2 v^T+\alpha_3 w^T$$

$$u=\begin{bmatrix}u_1\\ u_2\end{bmatrix} \quad v\in\begin{bmatrix}v_1\\ v_2\end{bmatrix} \quad w\in\begin{bmatrix}w_1\\ w_2\end{bmatrix}$$

V

$$\operatorname{span}(u_1,u_2,\dots,u_n)=\left\{\sum \alpha_i u_i \mid \alpha_i\in\mathbb{R}\ \ i=1\dots n\right\}$$

$$U=\begin{bmatrix}u_1^T\\ u_2^T\\ \vdots\\ u_n^T\end{bmatrix} \qquad =\left\{a^T U \mid a\in\mathbb{R}^n\right\}$$

$$a=\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix}\in\mathbb{R}^n \qquad a^T=\begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_n\end{bmatrix}$$

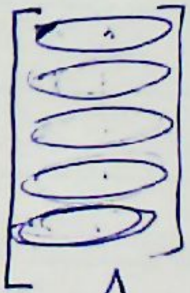
Row space of a matrix

The row space of a ~~matr~~ matrix $A \in \mathbb{R}^{m \times n}$

$= \text{span of rows of } A$

$$= \{ c^T A \mid c \in \mathbb{R}^m \} \quad [c_1 \; c_2 \; c_3 \; c_4 \; c_5]$$

$$\stackrel{?}{=} \{ A^T c \mid c \in \mathbb{R}^m \}$$



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