Linear Algebra for Computer Science

Lecture 5

Matrices & Linear Transformations

Review: column space, row space

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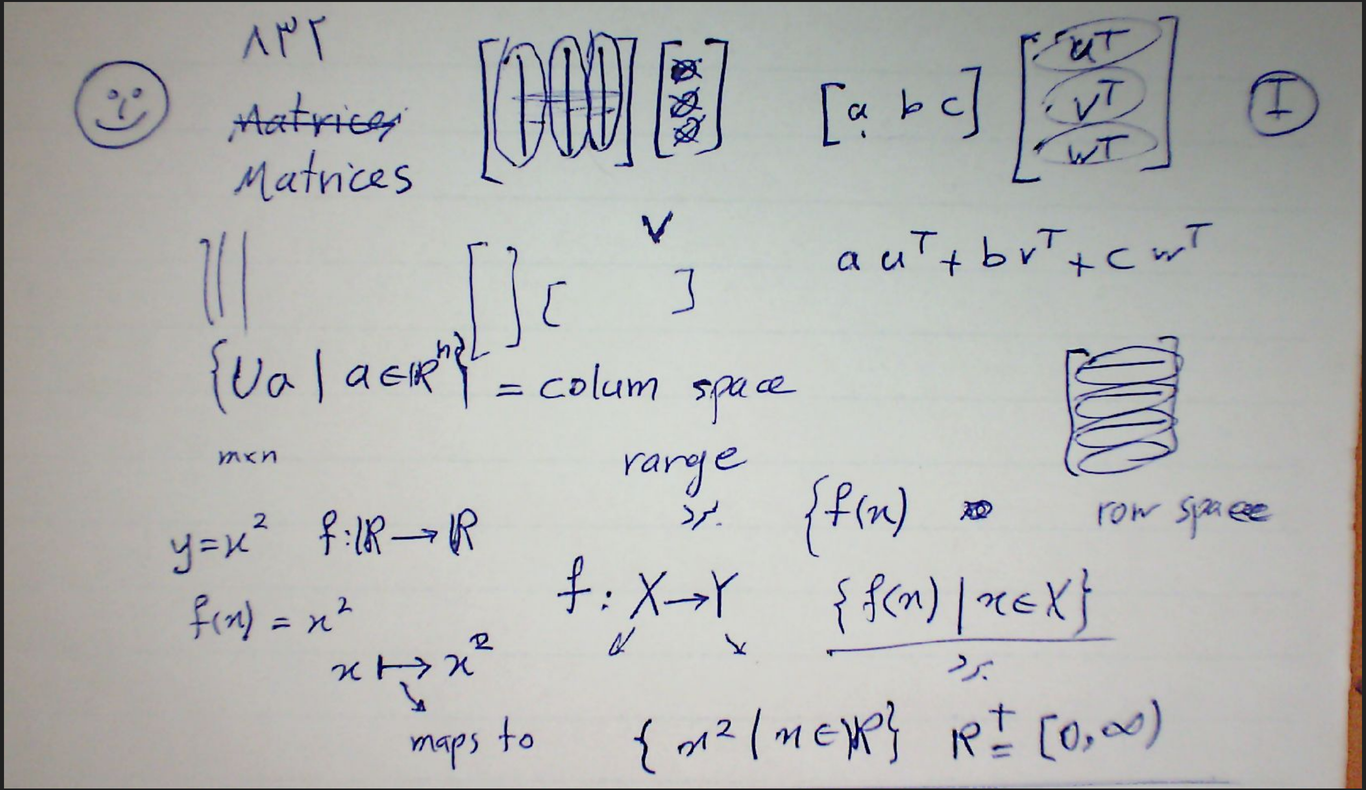

Matrices form a vector space?

the set of $m \times n$ matrices $\mathbb{R}^{m\times n}$
with real entries form a vetor space

$$A+B \quad \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \\ a_5+b_5 & a_6+b_6 \end{bmatrix}$$

element-wise

*

@

$$cA = c\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} = \begin{bmatrix} ca_1 & ca_2 \\ ca_3 & ca_4 \\ ca_5 & ca_6 \end{bmatrix}$$

8 Axioms ✓

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \quad \begin{bmatrix} x_7 \\ y_7 \end{bmatrix} \quad \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \end{bmatrix} \in \mathbb{R}^{n\times 2}$$

Shape models

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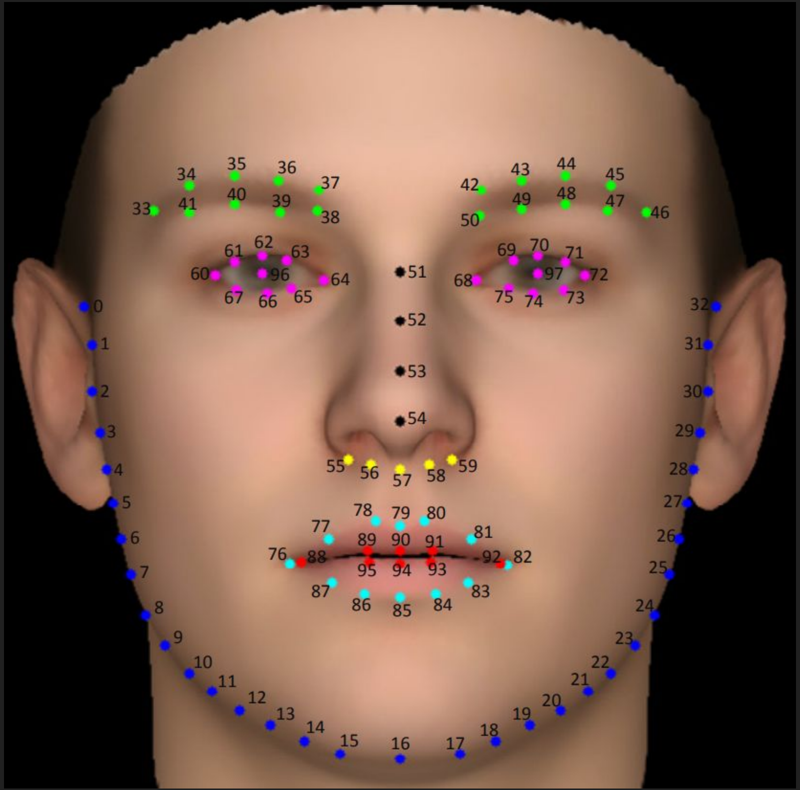

Shape models

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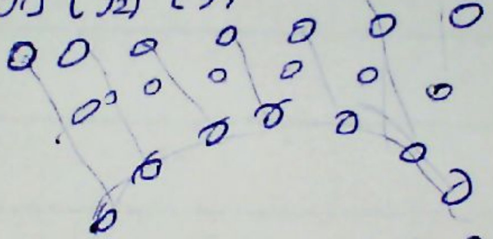
$$c\begin{bmatrix} a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \quad \begin{bmatrix} ca_3 & ca_4 \\ ca_5 & ca_6 \end{bmatrix}$$

8 Axioms ✓

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \quad \begin{bmatrix} x_7 \\ y_7 \end{bmatrix}$$

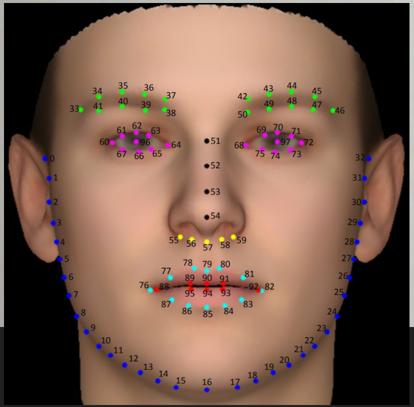
$$S_i = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \\ \vdots & \\ x_7 & y_7 \end{bmatrix} \in R^{n\times 2}$$

Shape Models



$$S_2 \in \mathbb{R}^{n\times 2}$$

$$0.5 * S_1 + 0.5 * S_2$$



Functions

- Also mappings, Transformations,
- What is a function?

functions / maps / tranformations

$f: X \longrightarrow Y$

II

$\mathrm{Domain}(f) = X$

$\mathrm{Codomain}(f) = Y$

$\mathrm{Range}(f) = \{ f(x) \mid x \in X \}$

Functions

functions / maps / tranformations

$f: X \longrightarrow Y$

$\mathrm{Domain}(f) = X$

$\mathrm{Codomain}(f) = Y$

$\mathrm{Range}(f) = \{ f(x) \mid x \in X \}$

f: one-to-one (injective) $\quad f(x) = f(y) \Longrightarrow x = y$

یک به یک

f: onto (surjective) $\quad \mathrm{Range}(f) = Y$

پوشا

$\forall y \in Y \ \exists x \in X : f(x) = y$

f: one-to-one & onto (bijective) $\Longrightarrow$ Invertible

یک به یک و پوشا $\Longrightarrow$ معکوس پذیر

Functions

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f: one-to-one & onto (bijective)

یک به یک و پوشا

$\Longrightarrow$ Invertible

معکوس پذیر

bijective $\exists g$ such that $g(f(x)) = x \quad \forall x \in X$

$\exists g \; Y \to X \qquad f(g(y)) = y \quad \forall y \in Y$

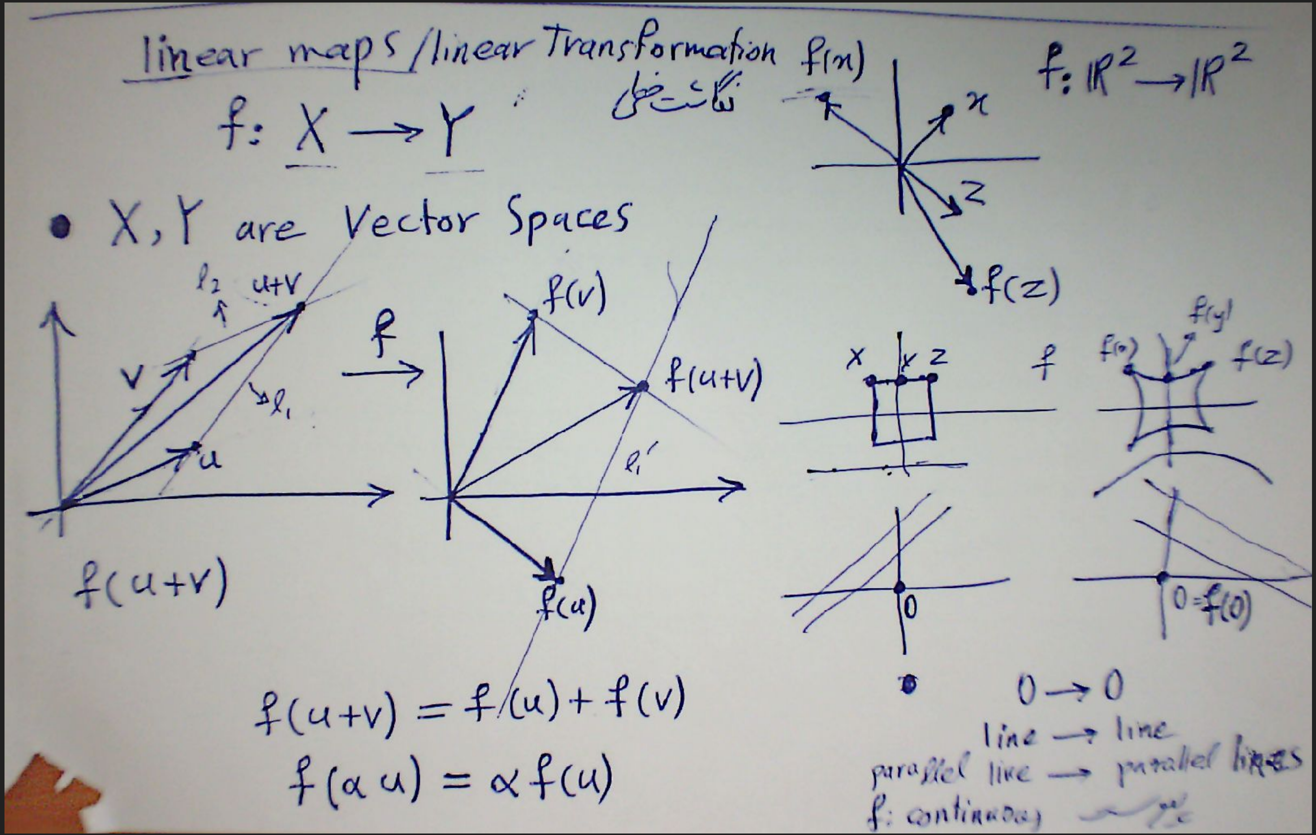
$g = f^{-1}$

Functions in linear algebra

- Here, we are interested in functions from a vector space V to a vector space U

  $(f: U \rightarrow V)$

Linear Transformations



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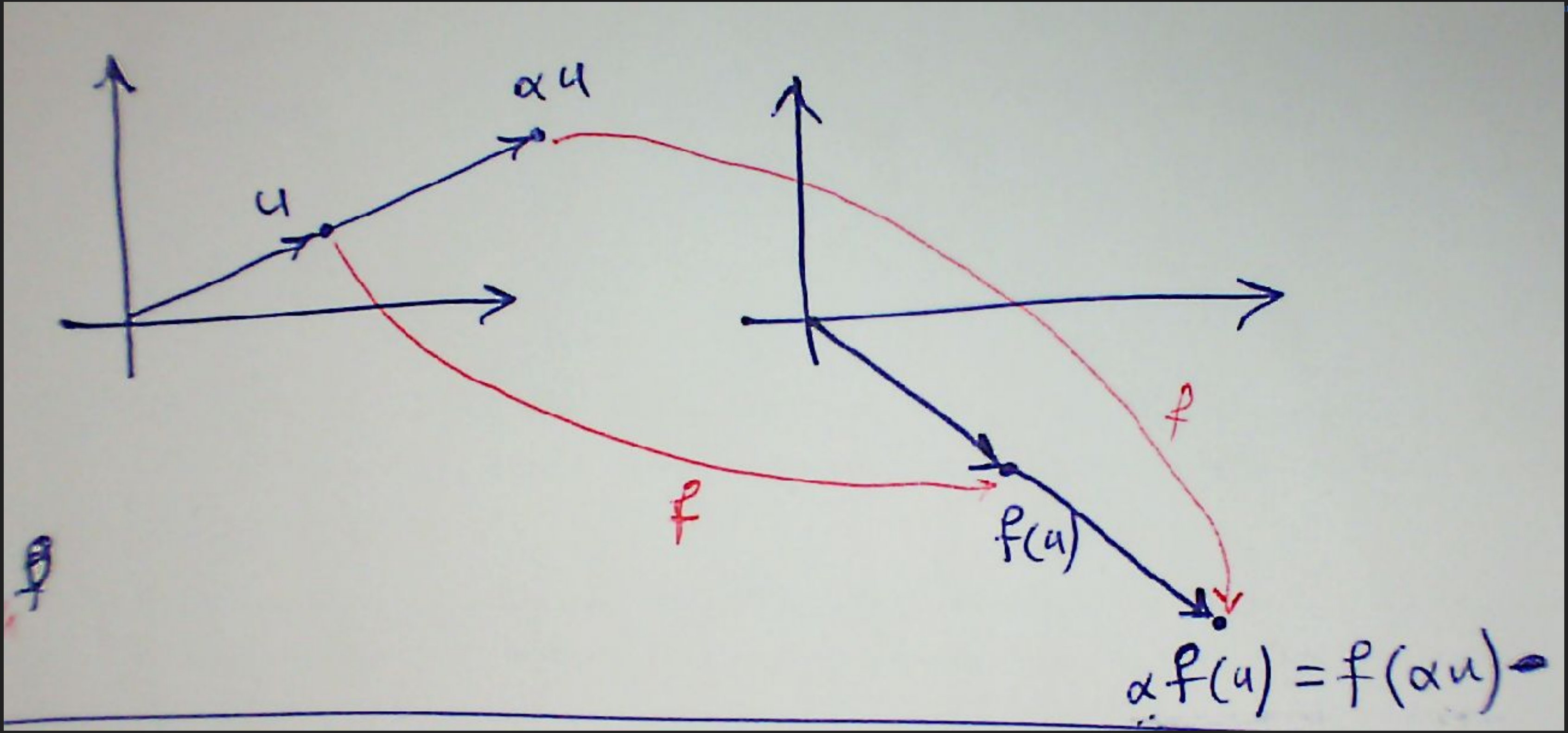
Linear Transformations



Linear Transformations

A linear map $f: V \longrightarrow U$

1. $f(u+v) = f(u)+f(v) \quad \forall u,v \in V$
2. $f(\alpha u) = \alpha f(u) \quad \forall u \in V, \alpha \in \mathbb{R}$

$\mathbb{C}$

$$1,2 \iff f(\alpha \underline{u} + \beta \underline{v}) = \alpha f(u) + \beta f(v) \quad \forall u,v \in V$$

$$f(\alpha u)+f(\alpha v) \qquad \forall \alpha,\beta \in \mathbb{R}$$

A lineap map preserves linear combinations

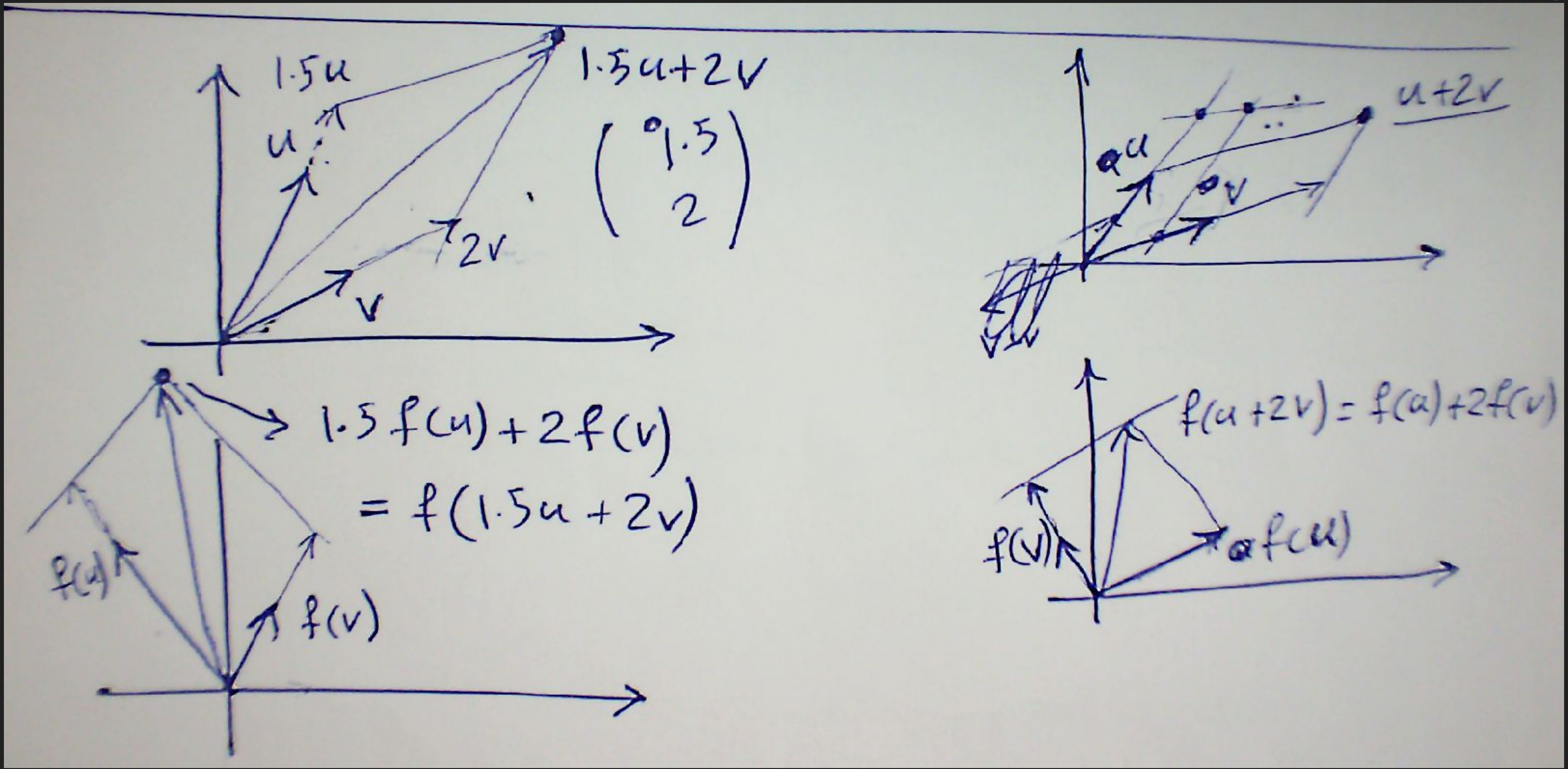
Linear Transformations

$f(u+v) = f(u) + f(v)$

$f(a\,u) = a\,f(u)$

does not matter if linear combination applied before or after transformation.

Linear Maps and Basis Vectors



Linear Maps and Standard Basis Vectors

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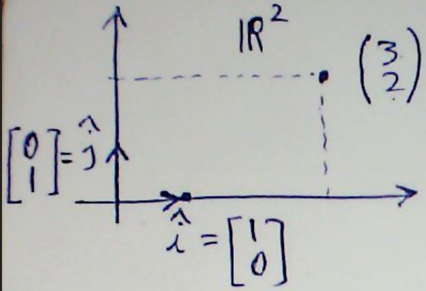
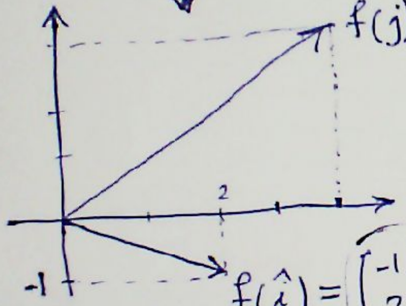


$$\begin{pmatrix}3\\2\end{pmatrix}=3\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}$$
$$=3\hat{i}+2\hat{j}$$

IV

$\downarrow$ f: linear



$$f(\hat{j})=f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{pmatrix}4\\3\end{pmatrix}$$

$$f(\hat{i})=\begin{pmatrix}-1\\2\end{pmatrix}$$

$$f\left(\begin{bmatrix}3\\2\end{bmatrix}\right)=f\left(3\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$=3f\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+2f\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$=3\begin{pmatrix}-1\\2\end{pmatrix}+2\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}5\\12\end{pmatrix}$$

$$f\left(\begin{bmatrix}x\\y\end{bmatrix}\right)\overset{?}{=}f\left(x\begin{bmatrix}1\\0\end{bmatrix}+y\begin{bmatrix}0\\1\end{bmatrix}\right)$$

Linear Maps and Standard Basis Vectors

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$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = f(x, y)$$

$$= x \begin{pmatrix} -1 \\ 2 \end{pmatrix} + y \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x\, f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + y\, f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$x \begin{pmatrix} -1 \\ 2 \end{pmatrix} + y \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$f\left(\begin{bmatrix} x & y \end{bmatrix}\right) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix}$$

Matrix Multiplication => linear map

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$f: \mathbb{R}^m \to \mathbb{R}^n$

$f(x) = Ax$

$A \in \mathbb{R}^{n \times m}$

$f(\alpha x + \beta y) = A(\alpha x + \beta y)$

$A(\alpha x) + A(\beta y) = \alpha Ax + \beta Ay = \alpha f(x) + \beta f(y)$

Every $f: \mathbb{R}^m \to \mathbb{R}^n$ in the form of $f(x) = Ax$ for $A \in \mathbb{R}^{n \times m}$ is linear.

linear map => matrix representation

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if $f: \mathbb{R}^m \to \mathbb{R}^n$ is linear the f can be represented as $f(x) = Ax$ for some $A \in \mathbb{R}^{n\times m}$.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$$

$$f(x) = f(x_1, x_2, \dots, x_m) = {}^{n}\left[f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) \quad f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) \cdots f\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right) \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

linear map => matrix representation

if $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is linear the f can be represented as $f(x) = Ax$ for some $A \in \mathbb{R}^{n \times m}$.

V

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m$$

$$f(x) = f(x_1, x_2, \dots, x_m) = \overset{n}{\left[\; f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) \quad f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) \cdots f\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right) \;\right]} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

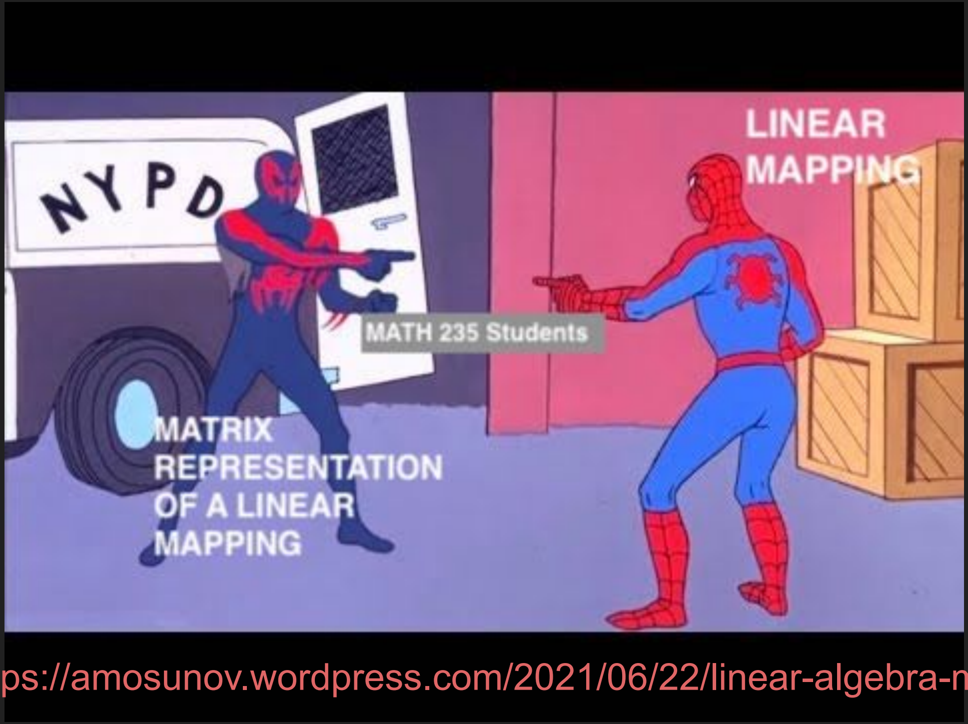
$$\underset{e_1}{\hat{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underset{e_2}{\hat{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \underset{e_3}{\hat{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

m standard basis vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad \dots \qquad e_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1 \underline{\underline{e_1}} + x_2 \underline{\underline{e_2}} + \cdots + x_m \underline{\underline{e_m}}$$

linear map <=> matrix representation



https://amosunov.wordpress.com/2021/06/22/linear-algebra-memes/

General finite dimensional vector spaces

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Let V be a vector space with finite no. of basis vectors $b_1, b_2, \dots, b_n$ (V is finite dimensioned)

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \text{ in } b_1, b_2, \dots, b_n.$$

$$v = v_1 b_1 + v_2 b_2 + \dots + v_n b_n$$

$f: V \longrightarrow U$ basis(U): $b'_1 \; b'_2 \dots b'_m$

$$f(b_1) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = u_1 b'_1 \; u_2 b'_2 \dots u_m b'_{(m)}$$