Linear Algebra for Computer Science

Lecture 6

Examples of Linear Maps
Composition of Linear Maps

Review

$$= \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}$$

f

$$u_2 = f(e_2) \qquad e_2 = \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix}$$

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$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_1 \qquad [x\ y]$$

$$\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = u_1 = f(e_1) \qquad f \qquad e_1$$

$$u_1\, x + u_2\, y = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} x + \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} y$$

$$[x\ y] \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Identity Transformation

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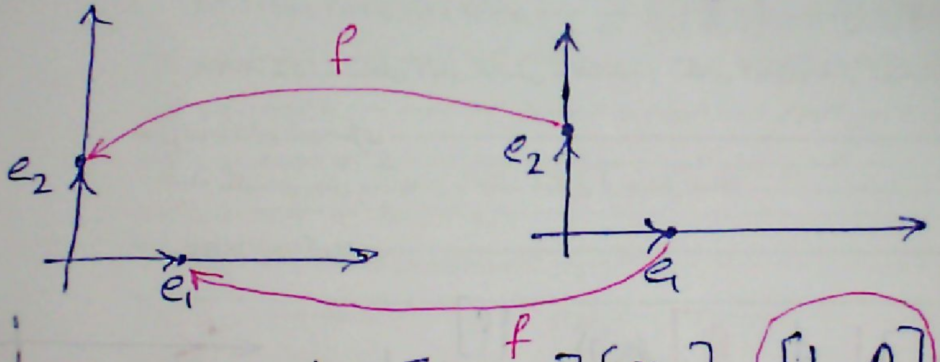
~~Ind.~~ Identity Map — نگاشت همانی

$$f(x) = x \qquad f(\alpha x+\beta y) = \alpha x+\beta y = \alpha f(x) + \beta f(y)$$



$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \underbrace{\begin{bmatrix} f(e_1) & f(e_2) \end{bmatrix}}_{M} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$M \searrow$ Transformation matrix

Identity Transformation

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \underbrace{\overset{f}{\begin{bmatrix} f(e_1) & f(e_2) \end{bmatrix}}}_{M} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$M \to$ Transformation matrix ~~Matrix~~

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

$$M = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{I = np.eye(n)} = I_n \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4 \to \text{identity matrix}$$

همانی

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Scaling

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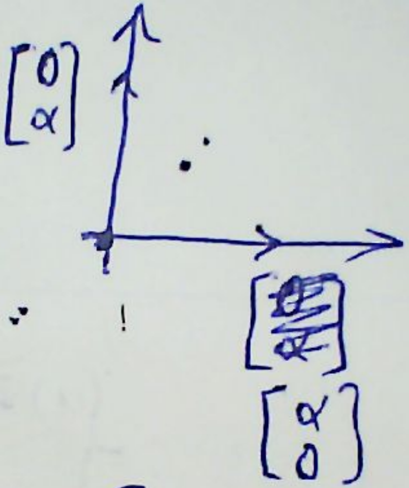
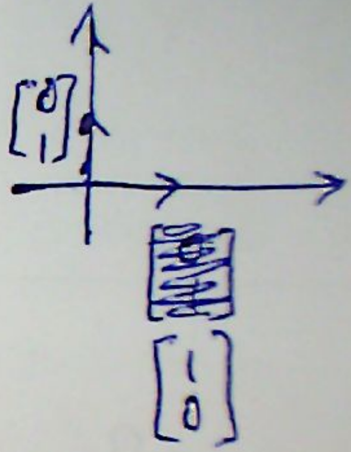
scaling (uniform)

$$f(\mathbf{x}) = \alpha \mathbf{x}$$

$$f(\mathbf{x}) = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \alpha I x = \alpha \mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}$$

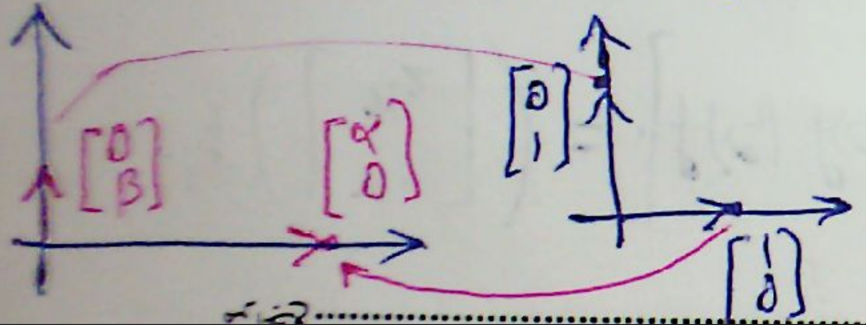




Non-uniform Scaling

non-uniform scaling

$$f\left(\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right)=\begin{bmatrix}\alpha x_1\\ \beta x_2\end{bmatrix}$$



$$f\left(\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right)=\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\alpha x_1\\ \beta x_2\end{bmatrix}$$

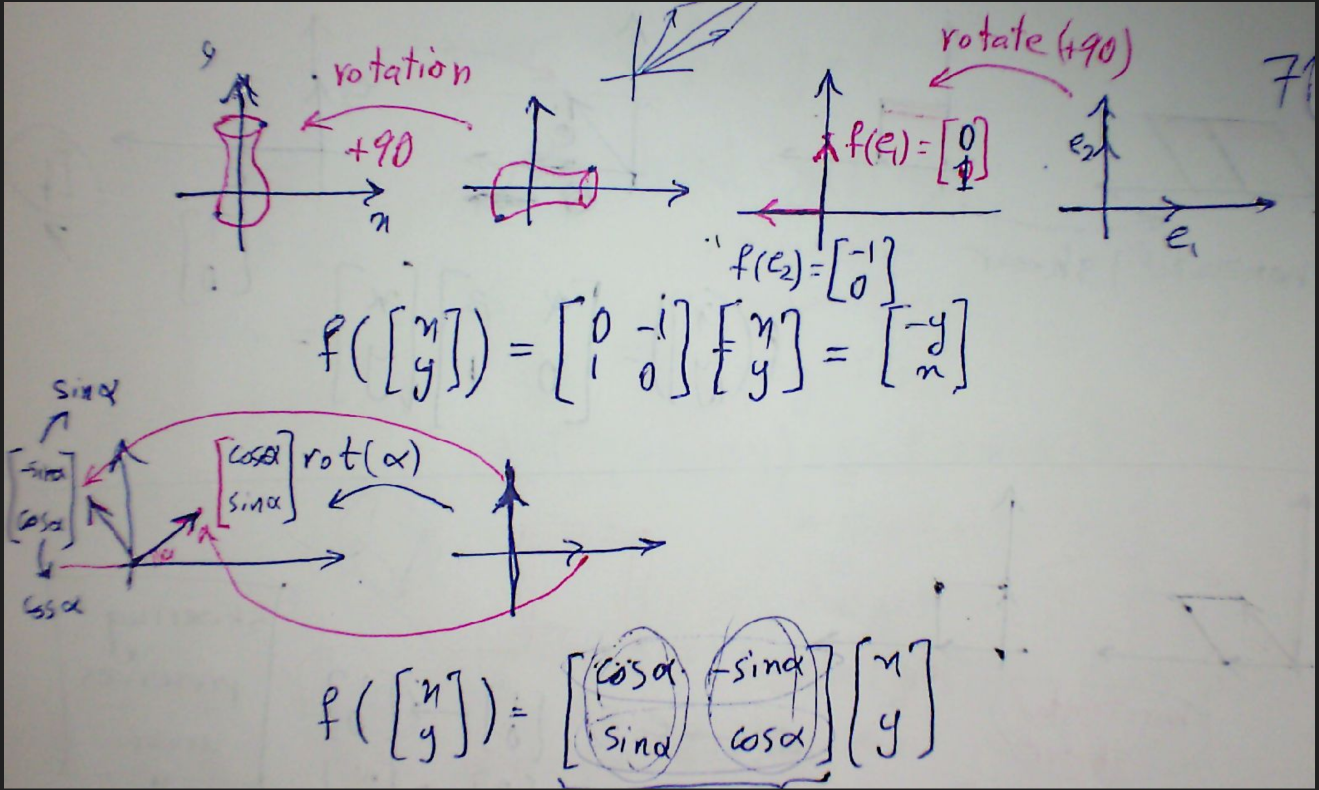
Non-uniform Scaling

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} \alpha_1 x_1 \\ \alpha_2 x_2 \\ \vdots \\ \alpha_n x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \alpha_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

ماتریس قطری

diagonal matrix

Rotation


rotation
+90
rotate (+90)
$f(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
e_2
e_1
$f(e_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$
$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$
sin α
$\begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}$
$\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ rot(α)
cos α
$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Rotation

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{R} \begin{bmatrix} x \\ y \end{bmatrix}$$

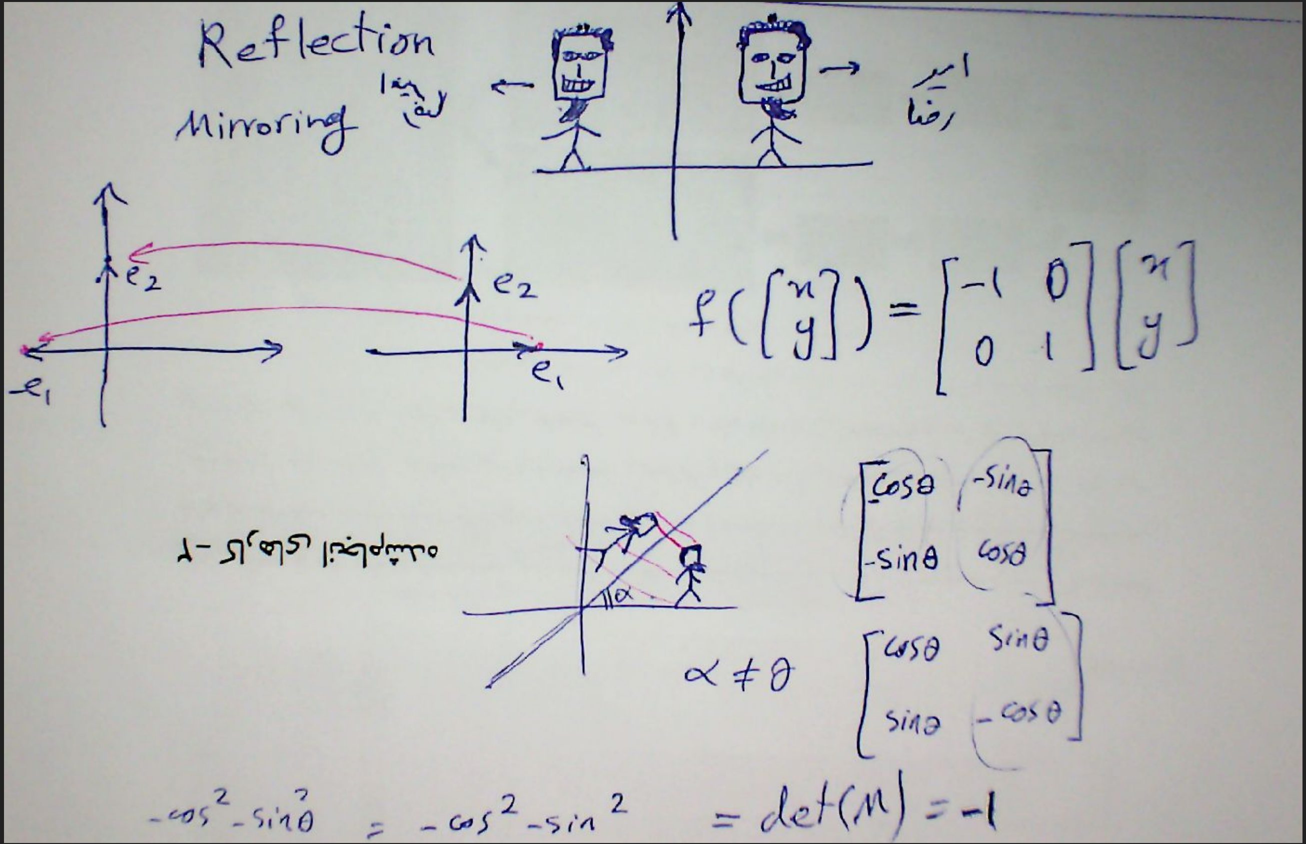
ماتریس چرخش Rotation matrix

orthogonal matrix $(R^{-1} = R^T)$

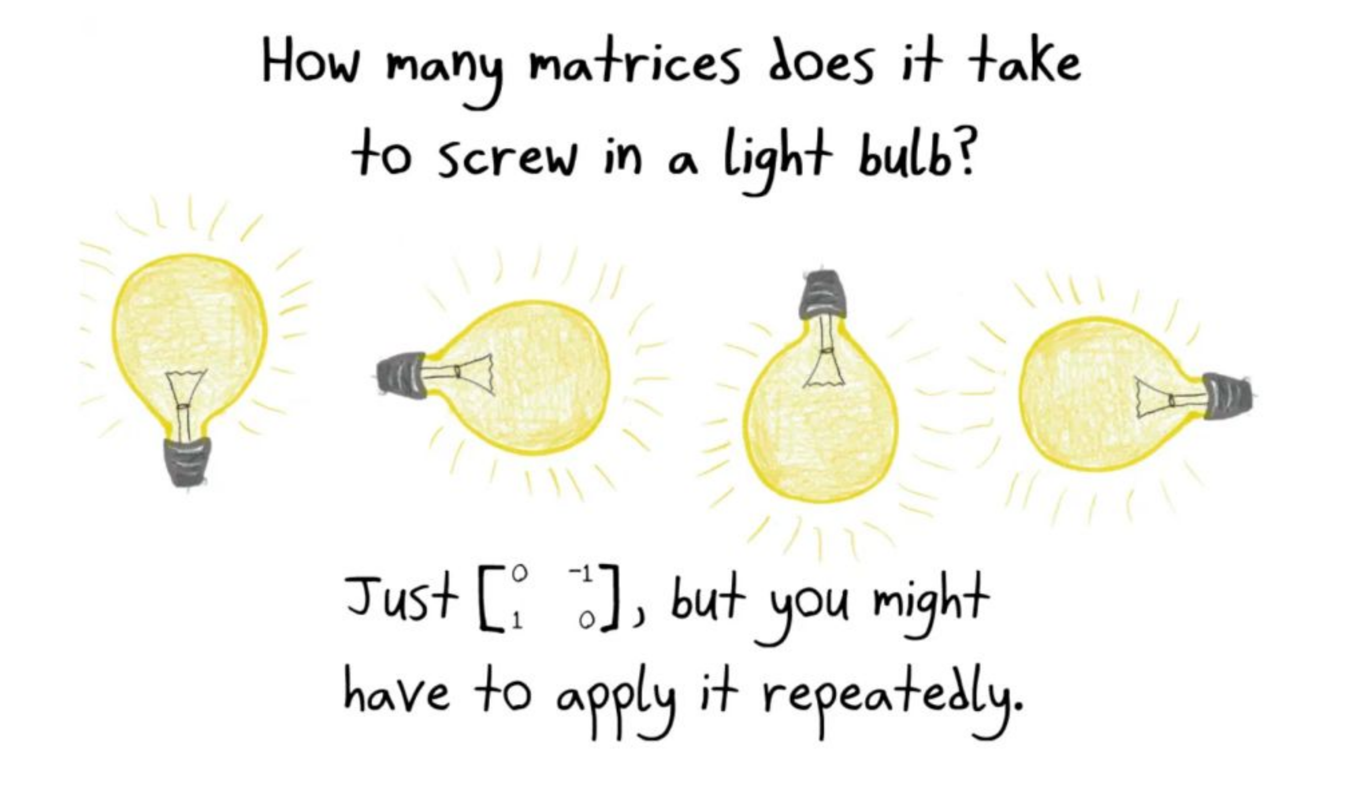
$$R^T R = R R^T = I$$

$$\det(R) = 1$$

Reflection


Reflection
Mirroring
امیر
رضا
e_2
$-e_1$
e_2
e_1
$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
α
$\begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$
$\alpha \neq \theta$
$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$
$-\cos^2 - \sin^2\theta = -\cos^2 - \sin^2 = \det(M) = -1$

Reflection



https://amosunov.wordpress.com/2021/06/22/linear-algebra-memes/

Shearing

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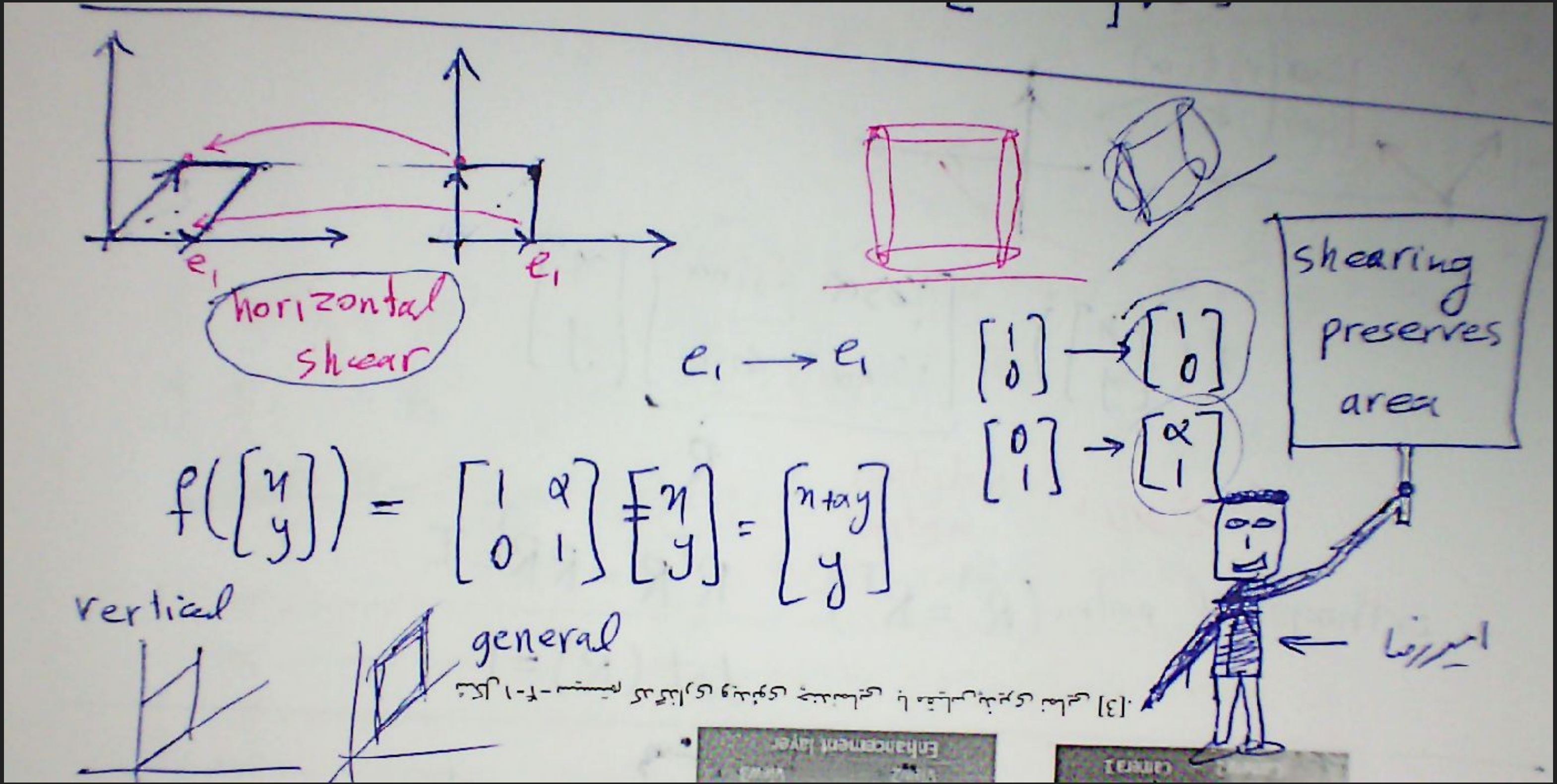

Degenerate Transformations

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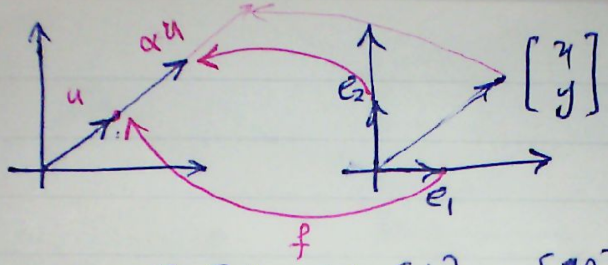

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = f\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$x f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + y f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$x \vec{u} + y(\alpha \vec{u})$$

$$x \vec{u} + (y\alpha)\vec{u}$$

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \underbrace{(x + y\alpha)}_{\beta}\vec{u} = \beta \vec{u}$$

degenerate transformation

Degenerate Transformations

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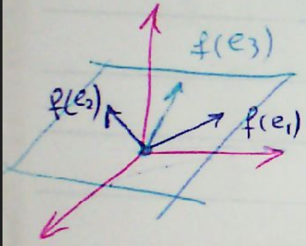
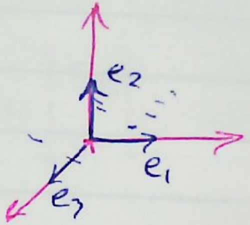
$$f\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=\begin{bmatrix}u & \alpha u\end{bmatrix} = \begin{bmatrix}p & \alpha p\\ q & \alpha q\end{bmatrix}$$

$$u=\begin{bmatrix}p\\q\end{bmatrix}$$

degerate





$$M=\begin{bmatrix} f(e_1) & f(e_2) & f(e_3)\end{bmatrix} =$$

$f(e_3) \rightarrow \alpha f(e_1) + \beta f(e_2)$

degenerate

Composition of Linear Maps

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f, g are linear $\qquad h = g \circ f$

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$f: \mathbb{R}^m \longrightarrow \mathbb{R}^n \qquad h(x) = g(f(x))$

$g: \mathbb{R}^n \longrightarrow \mathbb{R}^p$

ترکیب توابع
composition

$f: V \longrightarrow U$

$g: U \longrightarrow W$

$$(g \circ f)(x) = g(f(x))$$

$$\begin{aligned}(g \circ f)(\alpha x + \beta y) &= g(f(\alpha x + \beta y)) \\ &= g(\alpha \underline{f(x)} + \beta \underline{f(y)}) \\ &= \alpha g(f(x)) + \beta g(f(y))\end{aligned}$$

$$(g \circ f)(\alpha x + \beta y) = \alpha (g \circ f)(x) + \beta (g \circ f)(y)$$

Composition of Linear Maps

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$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad f(x) = Ax$

$g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad g(x) = Bx \qquad (f \circ g)(x) = Mx \qquad M = ?$

$M = BA$

$\cancel{(f \circ g)(x) = f} \; (g \circ f)(x) = g(f(x)) = g(Ax)$

$= B(Ax) \overset{?}{=} (BA)x$

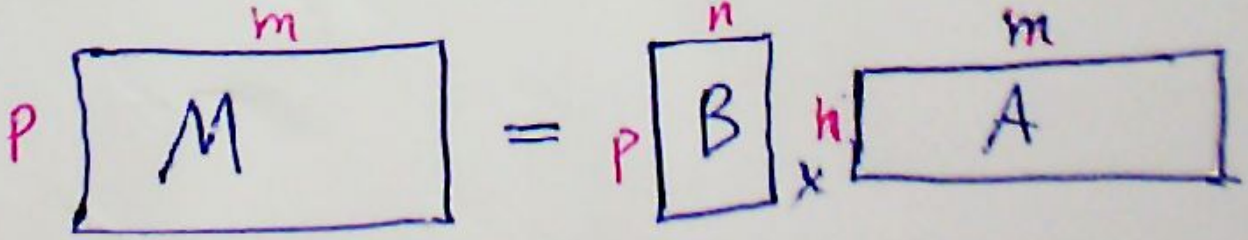
Composition of Linear Maps

ترکیب

$f: \mathbb{R}^m \longrightarrow \mathbb{R}^n \qquad f(x) = Ax \qquad A \in \mathbb{R}^{n\times m}$

$g: \mathbb{R}^n \longrightarrow \mathbb{R}^p \qquad g(y) = By \qquad B \in \mathbb{R}^{p\times n}$

$h = (g \circ f): \mathbb{R}^m \longrightarrow \mathbb{R}^p. \qquad h(x) = Mx \qquad M \in \mathbb{R}^{p\times m}$



Composition of Linear Maps

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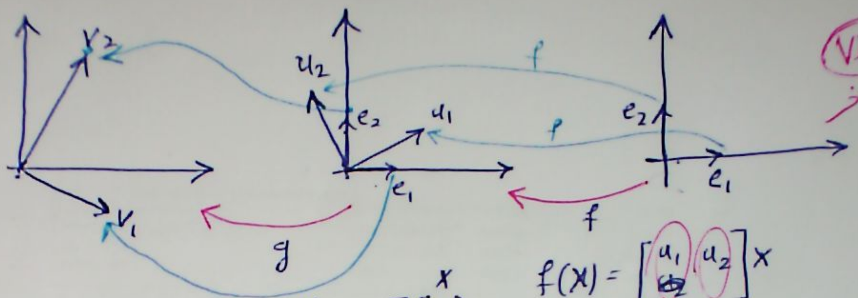

$$f(x) = \begin{bmatrix} u_1 & u_2 \end{bmatrix} x$$

2×2

$$g(x) = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$$f(x) = Ax$$

$$g(x) = Bx$$

$$(g \circ f)(x) = Mx \overset{?}{=} (BA)x$$

$$\vec{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}$$

$$\underline{B(Ax)}$$

$$(g \circ f)(e_1) = g(f(e_1)) = g(\vec{u}_1) = g\left(\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}\right)$$

$$g\left(u_{11}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_{21}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = u_{11}\vec{v}_1 + u_{21}\vec{v}_2$$

$$u_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} \qquad (g \circ f)(e_2) = u_{12}\vec{v}_1 + u_{22}\vec{v}_2$$

M

Composition of Linear Maps

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$$u_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} \qquad (g \circ f)(e_2) = u_{12}\vec{v}_1 + u_{22}\vec{v}_2$$

$$[u_1\ u_2] = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \qquad (g \circ f)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \overbrace{\begin{bmatrix} u_{11}\vec{v}_1 + u_{21}\vec{v}_2 & u_{12}\vec{v}_1 + u_{22}\vec{v}_2 \end{bmatrix}}^{M} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left[\, u_{11}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} + u_{21}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} \qquad u_{12}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} + u_{22}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} \,\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} u_{11}v_{11} + u_{21}v_{12} & u_{12}v_{11} + u_{22}v_{12} \\ u_{11}v_{21} + u_{21}v_{22} & u_{12}v_{21} + u_{22}v_{22} \end{bmatrix} = \cancel{\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}} \cancel{\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}}$$

$$= \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$