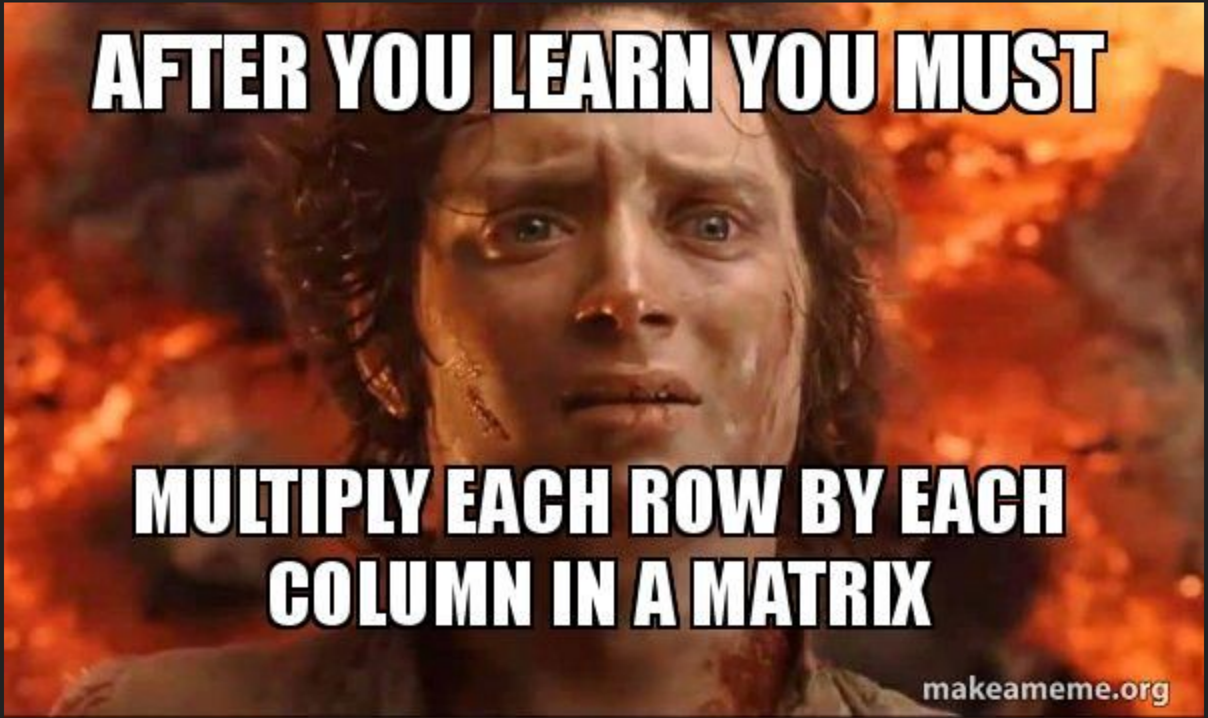
Linear Algebra for Computer Science

Lecture 7

Matrix Multiplication

Matrix Multiplication



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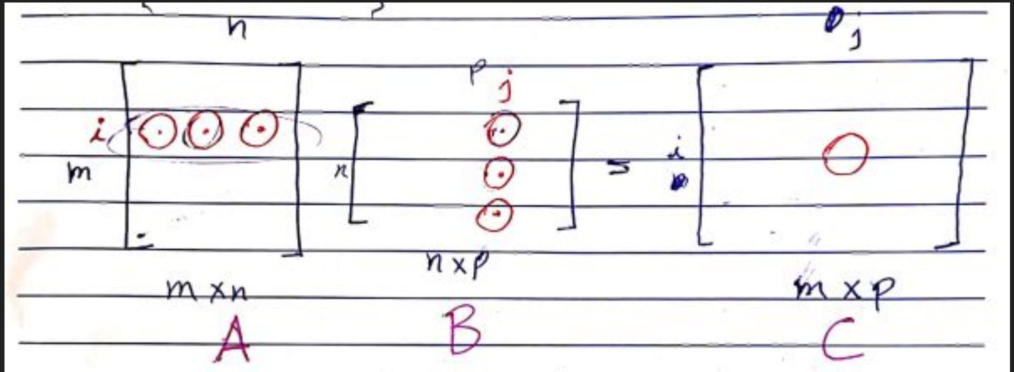
Matrix Multiplication



$$C_{ij} = A_{i1} B_{1j} + A_{i2} B_{2j} + \cdots + A_{in} B_{nj}$$

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

`np.sum(A[i,:] * B[:,j])`

Dot Product

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Inner product, dot product, scalar product

np.inner(u,v)

complex numbers?

$$\underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{u} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}}_{v} = ax + by + cz + dt$$

$$u \cdot v = \langle u, v \rangle$$

Dot Product as matrix product

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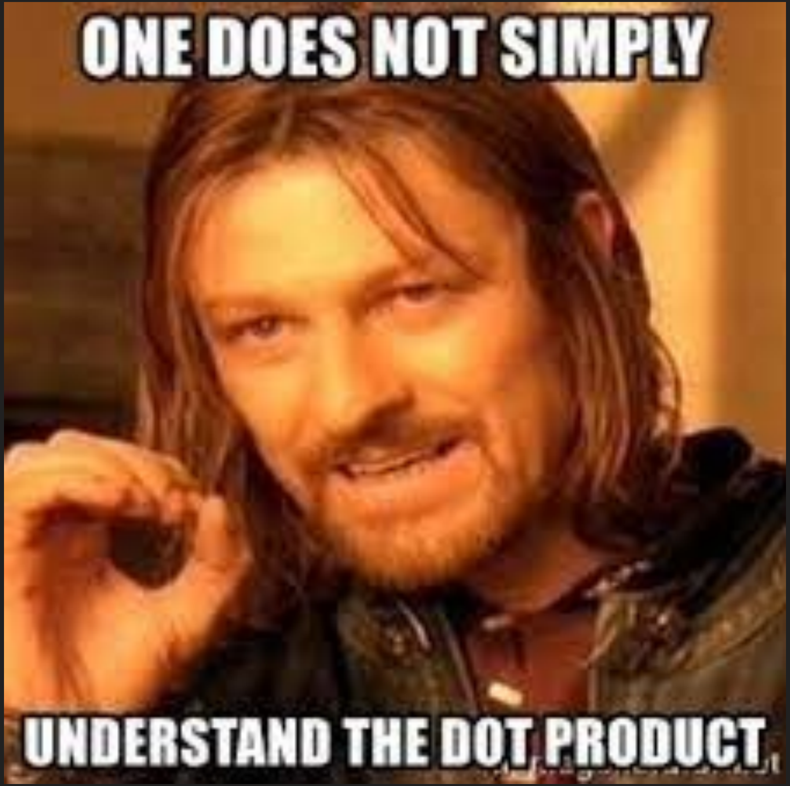
$$a = \underset{3\times 1}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} \qquad b = \underset{3\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

$$a^T b = \overbrace{\underset{1\times 3}{\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}}^{\text{ضرب ماتریسی}} = \underset{1\times 1}{\langle a, b \rangle}$$

$$b^T a = a^T b = \langle a, b \rangle$$

Inner Product



Inner Product

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$$\langle u+w, v\rangle = \langle u, v\rangle + \langle w, v\rangle$$

$$\left\langle \begin{bmatrix} u_1+w_1 \\ u_2+w_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle =$$

$$\langle \alpha u, v\rangle = \alpha \langle u, v\rangle$$

$$\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$$

$$\langle u, \alpha v\rangle = \alpha \langle u, v\rangle$$

$$\langle u, v\rangle = f(u, v)$$

$$\cancel{f(u,v)=} \quad f(u+w, v) = f(u, v) + f(w, v)$$

$$f(\alpha u, v) = \alpha f(u, v)$$

Inner Product

$$f(x,y) = f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = xy$$

$$f\left(\alpha \begin{bmatrix} x \\ y \end{bmatrix}\right) = f\left(\begin{pmatrix} \alpha x \\ \alpha y \end{pmatrix}\right) = (\alpha x)(\alpha y) = \alpha^2 xy$$

$$\begin{bmatrix} u \\ v \end{bmatrix} \qquad f\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ t \end{bmatrix}\right) = (x+z)(y+t) \neq xy + zt$$

$$f(u,v) = f\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = f\left(\begin{bmatrix} \alpha u \\ v \end{bmatrix}\right) = \alpha f\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$$

$$f\left(\begin{bmatrix} u+w \\ v \end{bmatrix}\right) = f\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) + f\left(\begin{bmatrix} w \\ v \end{bmatrix}\right)$$

$f(x,y)$ is linear in x } bilinear
is linear in y }
is not linear in $\begin{pmatrix} x \\ y \end{pmatrix}$

$\langle u+w, v \rangle$

General vector spaces: Inner product space

① $\begin{cases} \langle \alpha u, v \rangle = \alpha \langle u, v \rangle \\ \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \end{cases}$

② $\langle u, v \rangle = \langle v, u \rangle$

③ $\begin{cases} \langle u, u \rangle > 0 & u \neq 0 \\ \langle u, u \rangle = 0 & u = 0 \end{cases}$

$\langle 0, u \rangle = \langle v - v, u \rangle$

$f(u, v) \quad V \times V \longrightarrow \mathbb{R}$

Matrix Multiplication in terms of inner products

IV

$$
\begin{array}{cc} & n \\ m & \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_m^T \end{bmatrix} \end{array}
\begin{array}{c} p \\ \begin{bmatrix} c_1 & c_2 & \cdots & c_p \end{bmatrix} \end{array}
=
\begin{array}{cc} & j \\ i & \begin{bmatrix} & & \\ & r_i^T c_j & \\ & & \end{bmatrix} \end{array}
$$

$$A \qquad\qquad B \qquad\qquad M$$

$$M_{ij} = \langle r_i , c_j \rangle = r_i^T c_j$$

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Matrix multiplications in terms of matrix-vector product

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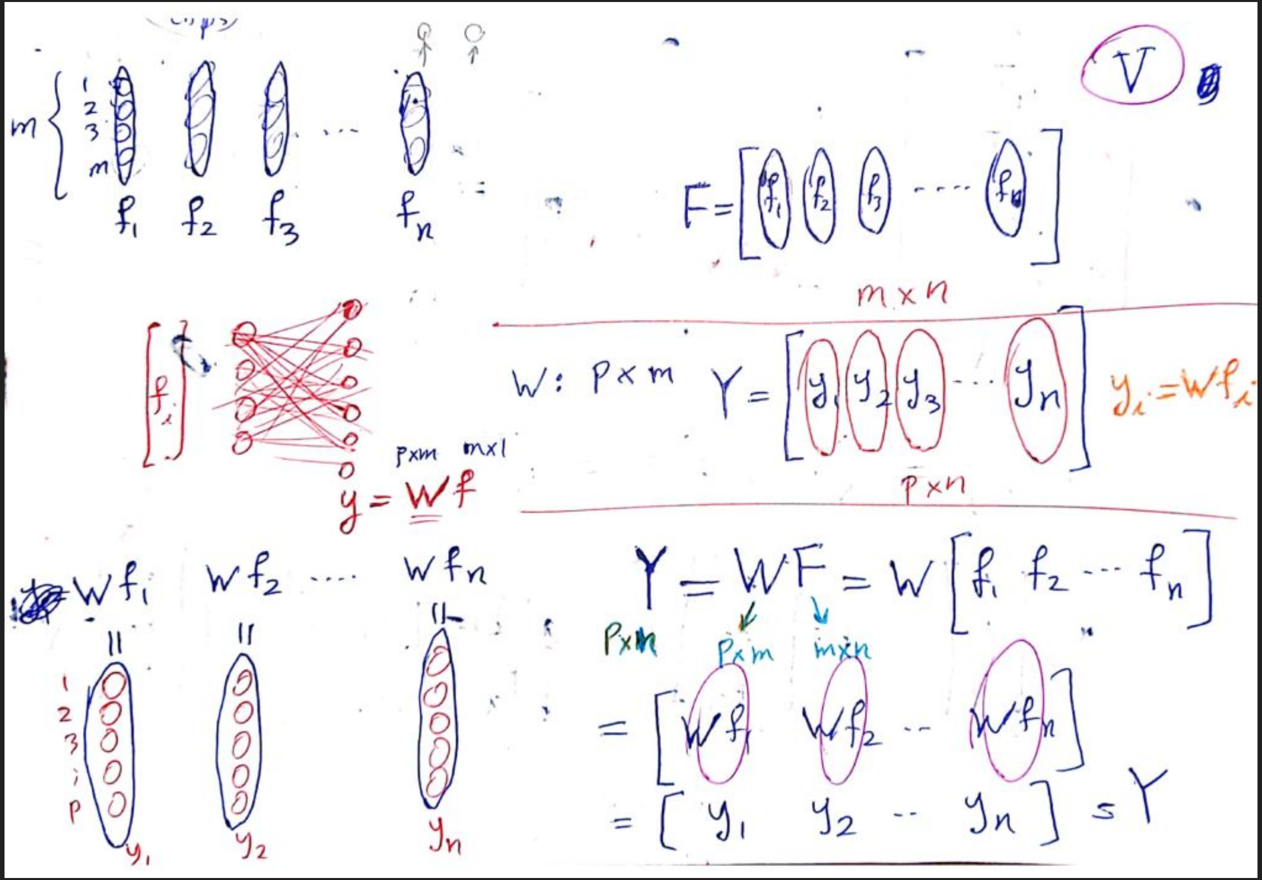


Transforming a bunch of points
data points as columns of a matrix

$$\underbrace{\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}}_{\substack{A \\ m\times p}} \underbrace{\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}}_{\substack{B \\ p\times n}} = \underbrace{\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}}_{\substack{C \\ m\times n}}$$

VI

$$\boxed{c_i = A b_i}$$

$$\begin{bmatrix} a_1^T \\ \vdots \end{bmatrix} \begin{bmatrix} \; \end{bmatrix}$$

Transforming a bunch of points
data points as rows of a matrix

$$\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_m^T \end{bmatrix}$$

A $\quad m \times p$

B $\quad p \times n$

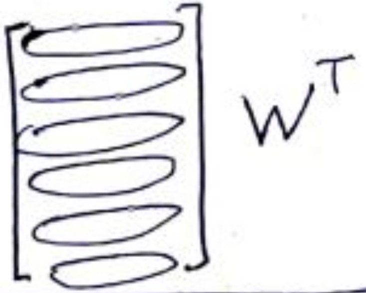
C $\quad m \times n$

$$\boxed{c_i^T = a_i^T B}$$

be careful about linear transformations on row vectors!

$$R\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$R\left[\begin{bmatrix}x_1\\y_1\end{bmatrix}\ \begin{bmatrix}x_2\\y_2\end{bmatrix}\ \begin{bmatrix}x_3\\y_3\end{bmatrix}\right]$$



$$\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\\ \vdots & \\ x_n & y_n\end{bmatrix}R^T = \begin{bmatrix}x_1 & y_1\\ x_2 & y_2\\ \vdots & \\ x_n & y_n\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

be careful about linear transformations on row vectors!

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$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \\ x_n & y_n \end{bmatrix} R^T = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \\ x_n & y_n \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$R \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$R \left[\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} \right]$$

Outer Product

np.outer(u,v)

u @ v.T

How many independent columns?

How many independent rows?

outer(u,v) = outer(v,u).T

complex numbers?

Outer Product

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Outer product

VII

$$u = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad v = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

$u \in \mathbb{R}^3 \qquad v \in \mathbb{R}^4$

$$\begin{array}{cc} & \begin{array}{cccc} x & y & z & t \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} ax & ay & az & at \\ bx & by & bz & bt \\ cx & cy & cz & ct \end{bmatrix} \end{array} = \begin{bmatrix} a(x, y, z, t) \\ b(x, y, z, t) \\ c(x, y, z, t) \end{bmatrix}$$

3 × 4

$W \in \mathbb{R}^{3\times 4}$

$$\boxed{W = u \otimes v}$$

$\in \mathbb{R}^{3\times 1}$

Outer Product

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$$\boxed{W = u \otimes v}$$

$u \in \mathbb{R}^{3\times1}$

$\therefore v \otimes u = W^T$

$$W = \begin{bmatrix} x\begin{bmatrix}a\\b\\c\end{bmatrix} & y\begin{bmatrix}a\\b\\c\end{bmatrix} & z\begin{bmatrix}a\\b\\c\end{bmatrix} & t\begin{bmatrix}a\\b\\c\end{bmatrix}\end{bmatrix} = \begin{bmatrix} x u & y u & z u & t u\end{bmatrix}$$

$$W = \underset{3\times1}{\begin{bmatrix}a\\b\\c\end{bmatrix}} \underset{1\times4}{\begin{bmatrix}x & y & z & t\end{bmatrix}} = \overset{3\times1}{u}\,\overset{1\times4}{v^T} = \overset{3\times4}{W}$$

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$u \in \mathbb{R}^{3\times1}$ $\quad v \in \mathbb{R}^{4\times1}$

Inner product vs outer product

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inner product $u^T v \in \mathbb{R}$

$u, v \in \mathbb{R}^{n\times 1}$

$u \cdot v = \langle u, v \rangle = u^T v$

outer prodct $u v^T \in \mathbb{R}^{m\times n}$

$u \in \mathbb{R}^{m\times 1}$ $\quad v \in \mathbb{R}^{n\times 1}$

$u \otimes v = u v^T$

Two ways of looking at matrix product

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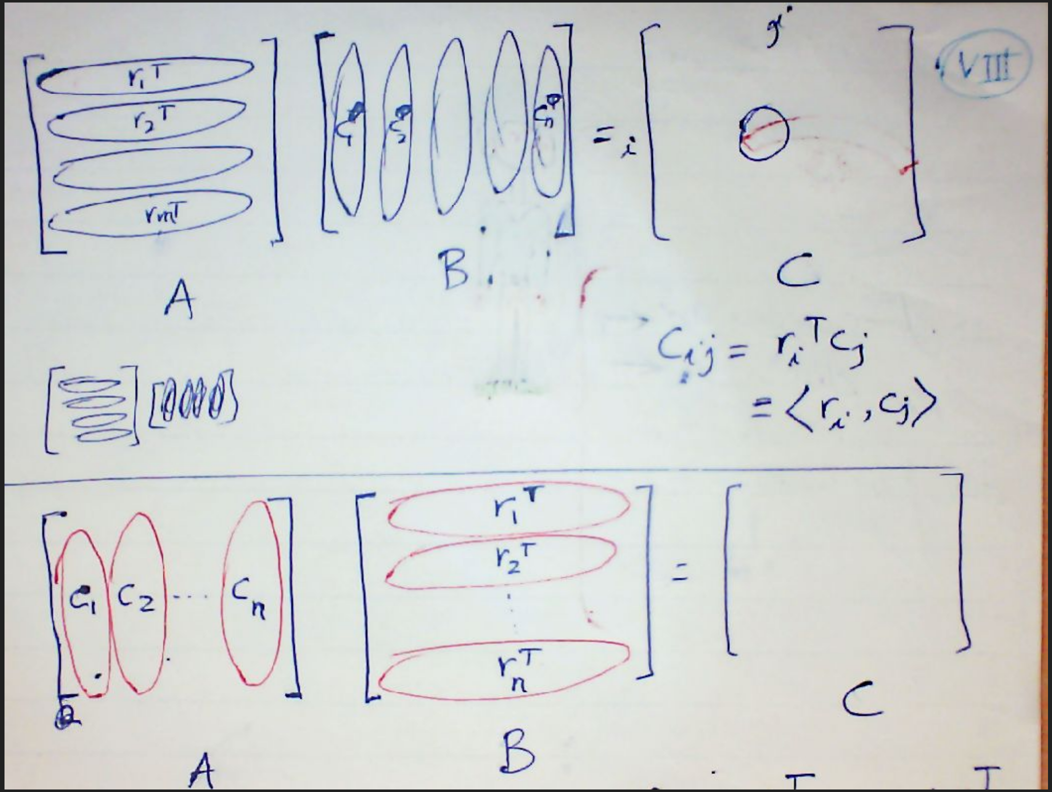

Matrix Multiplication in terms of outer products

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$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} ax+bz & ay+bt \\ cx+dz & cy+dt \\ ex+fz & ey+ft \end{bmatrix}$$

$$\begin{bmatrix} ax & ay \\ cx & cy \\ ex & ey \end{bmatrix} + \begin{bmatrix} bz & bt \\ dz & dt \\ fz & ft \end{bmatrix}$$

$$= \begin{bmatrix} a \\ c \\ e \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} + \begin{bmatrix} b \\ d \\ f \end{bmatrix} \begin{bmatrix} z & t \end{bmatrix}$$

$$= \begin{bmatrix} a \\ c \\ e \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b \\ d \\ f \end{bmatrix} \otimes \begin{bmatrix} z \\ t \end{bmatrix}$$

Matrix Multiplication in terms of outer products

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$$\underset{A}{\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}} \; \underset{B}{\begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_n^T \end{bmatrix}} = \underset{C}{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}$$

$$C = c_1 r_1^T + \cdots + c_n r_n^T$$

$$C = \sum_{i=1}^{n} c_i \, r_i^T$$

$$= \sum_{i=1}^{n} c_i \otimes r_i$$

Example

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$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix}}_{2\times 3} + \underbrace{\begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} d & e & f \end{bmatrix}}_{2\times 3}$$

$$+ \underbrace{\begin{bmatrix} 3 \\ 6 \end{bmatrix} \begin{bmatrix} g & h & i \end{bmatrix}}_{2\times 3}$$

Block-wise multlipication

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$$\underset{m\times n}{\overset{n}{A}}\ \underset{n\times p}{B} = \begin{bmatrix} \underset{m\times n_1}{A_1} & \underset{m\times n_2}{A_2}\end{bmatrix}\begin{bmatrix} B_1 \\ B_2\end{bmatrix} \begin{matrix} \to n_1\times p \\ \to n_2 \times p\end{matrix} = \underset{m\times p}{A_1B_1} + \underset{m\times p}{A_2B_2}$$

$$n_1+n_2=n$$

Block-wise multlipication

$$A \quad B = \overset{n}{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}} n \, B = \begin{bmatrix} A_1 B \\ A_2 B \end{bmatrix}$$

Block-wise multlipication

$$\begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \begin{array}{cc} \begin{array}{cc} p_1 & p_2 \end{array} & \\ \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} & \begin{array}{c} n_1 \\ n_2 \end{array} \end{array}$$

$$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$