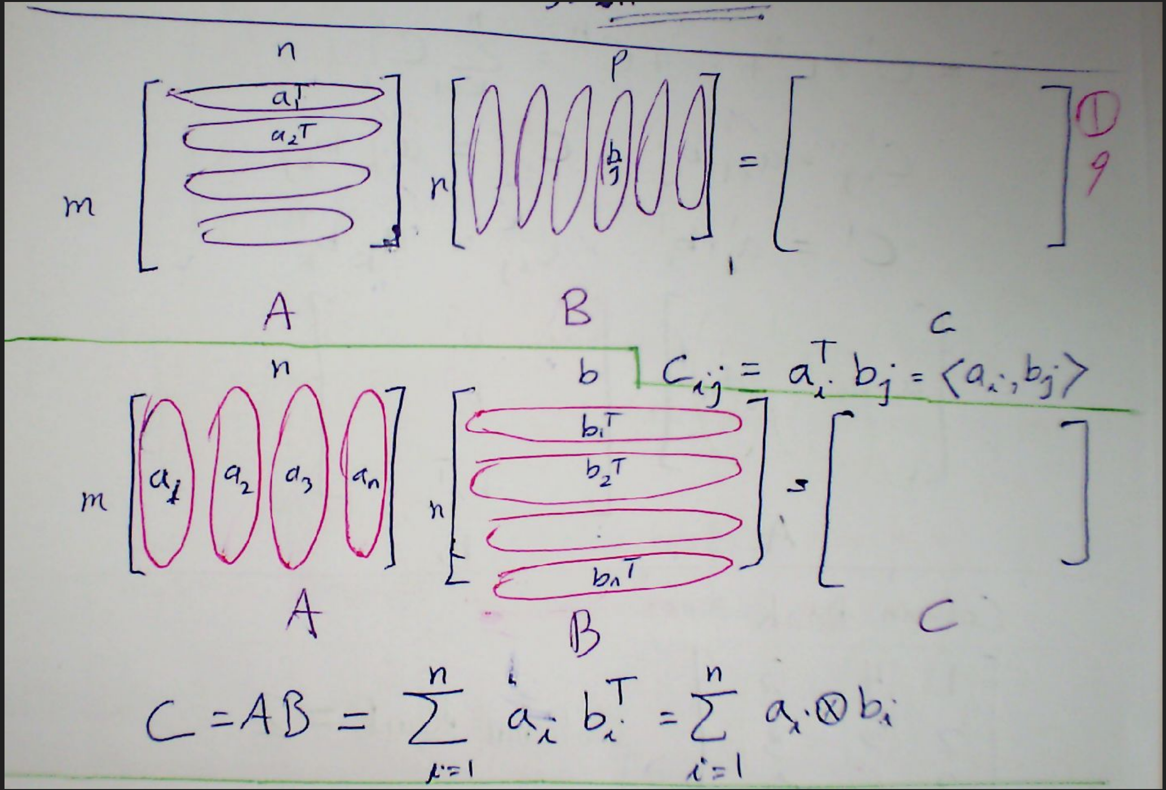
Linear Algebra for Computer Science

Lecture 8

Matrix Rank, Linear Equations

Review: Matrix Multiplication

K. N. Toosi
University of Technology



Review: Outer Product

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \qquad \vec{a} \otimes \vec{b} = \vec{a}\vec{b}^T = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \end{bmatrix}$$

$$a \otimes b = ab^T = \begin{bmatrix} b_1\vec{a} & b_2\vec{a} & b_3\vec{a} & b_4\vec{a} \end{bmatrix} = \begin{bmatrix} a_1 b^T \\ a_2 b^T \\ a_3 b^T \end{bmatrix}$$

(at most) 1 independent column 1 independent row

Review: Outer Product

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University of Technology

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9

$$A = [a_{ij}]_{m\times n} \qquad B = [b_{ij}]_{n\times p} \qquad C = [c_{ij}]_{m\times p}$$

$$\cancel{a_{ij}} = \boxed{c_{ij}} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$C^1, C^2, \ldots, C^n \in \mathbb{R}^{m\times p}$$

$$C = C^1 + C^2 + \cdots + C^n = \sum_{k=1}^{n} C^i$$

$$C^1_{ij} = a_{i1} b_{1j} \qquad C^k_{ij} = a_{ik} b_{kj}$$

$$C^1 = a_1 b_1^T \qquad C^k_{ij} = a_k b_k^T$$

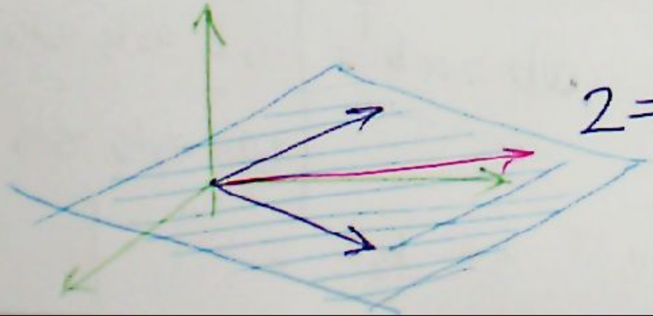
$$\underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}}_{A} \qquad \underbrace{\begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{bmatrix}}_{B}$$

Column Rank

Column Rank

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 3 \\ 3 & 6 & 3 \end{bmatrix}$$

Column Rank $= 2$



$2 = \dim(\text{column space})$

Column Rank and Row Rank

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \\ 3 & -6 \\ 4 & -8 \end{bmatrix}$$

column rank = 1

row rank = 1

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 0 & 5 \\ 3 & 0 & 6 \end{bmatrix}$$

column rank = 2

row rank = 2

Column Rank and Row Rank

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

column rank = 3

row rank = 3

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

colum rank = 0

row rank = 0

Column Rank and Row Rank

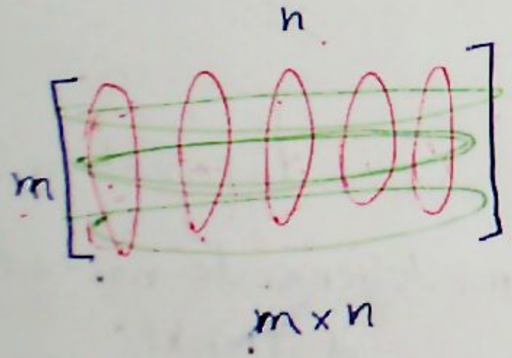
$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \vec{a} \otimes \vec{b} = \vec{a}\,\vec{b}^T = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \end{bmatrix}$$

$$a \otimes b = a\,b^T = \begin{bmatrix} b_1\vec{a} & b_2\vec{a} & b_3\vec{a} & b_4\vec{a} \end{bmatrix} = \begin{bmatrix} a_1\,b^T \\ a_2\,b^T \\ a_3\,b^T \end{bmatrix}$$

(at most) 1 independent column 1 independent row

$$\text{RowRank}(a\,b^T) = \text{ColumnRank}(a\,b^T) \leq 1$$

Column Rank and Row Rank



$\le m$

column rank = dim(column space) $\le n$

row rank = dim(row space) $\le m$

$\le n$

column rank $\le \min(m, n)$

row rank $\le \min(m, n)$

Column Rank and Row Rank

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$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 5 & 1 \\ 3 & 4 & 7 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}}_{2\times 4}$$

Matrix Factorization

Row Rank = Col Rank

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

```
L = []
for i = 1 .. n
    if c_i ∉ span(c_1, ..., c_{i-1})      span(L)
        L.append(c_i)
```

Column Rank and Row Rank

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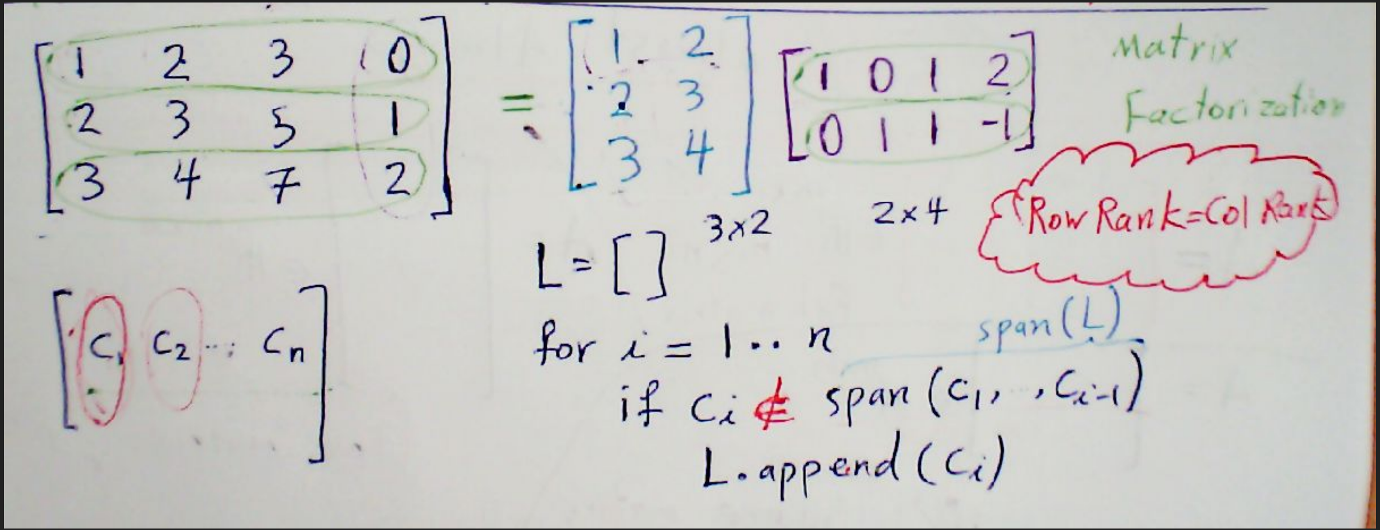

Column Rank = Row Rank

Column Rank and Row Rank

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$$\begin{array}{cc} & n \\ m & \begin{bmatrix} & & A & & \end{bmatrix} \end{array} = \begin{array}{cc} & r \\ m & \begin{bmatrix} \; & \; & \; \end{bmatrix} \end{array} \begin{array}{cc} & n \\ r & \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} \end{array}$$

IV

a basis for column space. a basis for row space

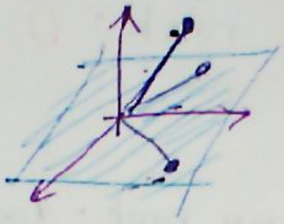
column rank $= r \leq \min(m,n)$

$$\boxed{\begin{aligned} \text{Row Rank}(A) &= \text{Column Rank}(A) \\ &\overset{\text{def}}{=} \text{Rank}(A) \;.\; \text{رتبه} \end{aligned}}$$

$$\text{Rank}(A) = \dim(\text{Colspace}(A)) = \dim(\text{RowSpace}(A))$$

"Most" matrices have full rank

$$\begin{array}{cc} & 3 \\ 3 & \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \\ & A \end{array}$$

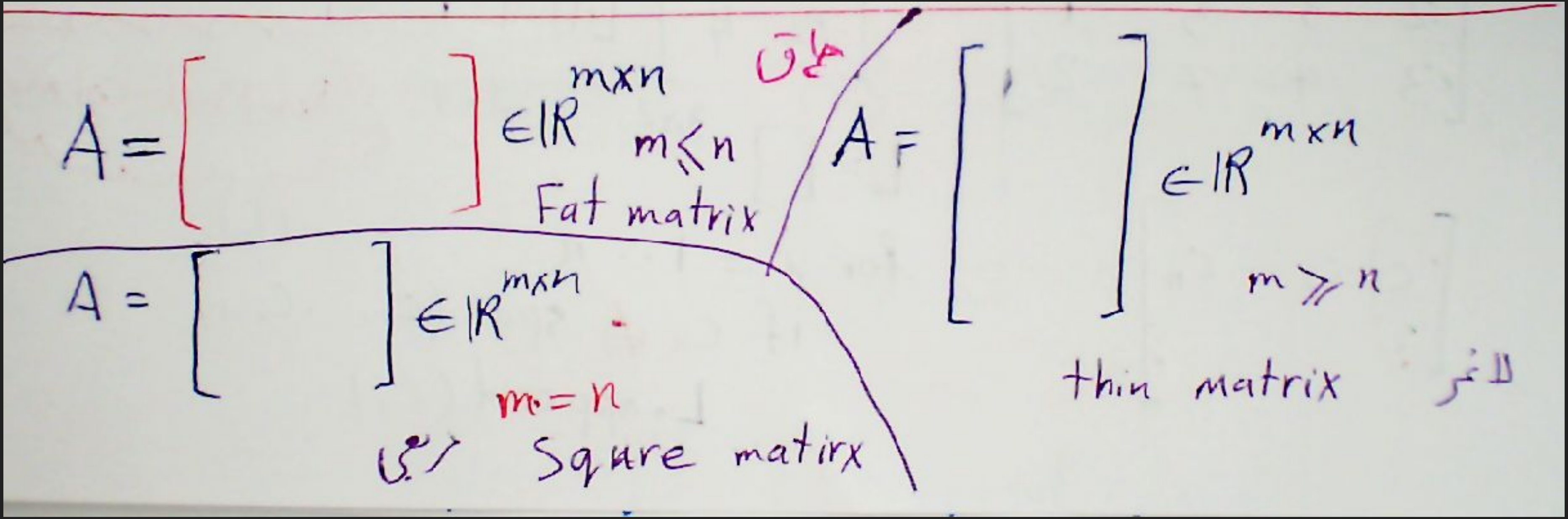


if A is a random 3×3 matrix (uniform distribution
nondegenerate normal
distribution)

$\text{rank}(A) = 3$ <u>with probability = 1</u>

<u>almost Always</u>

thin and fat matrices



full-rank and rank-deficient

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$A \in \mathbb{R}^{m \times n}$ $\qquad \mathrm{rank}(A) \leq \min(m,n)$

$$\begin{cases} \mathrm{rank}(A) = \min(m,n) & \text{full-rank} \\ \mathrm{rank}(A) < \min(m,n) & \text{rank-deficient} \end{cases}$$

رتبه کامل

رتبه ناقص

A is full-rank

A has full rank

full-rank and rank-deficient

$A \in \mathbb{R}^{m \times n}$ $\qquad \text{rank}(A) \leq \min(m,n)$

$$\begin{cases} \text{rank}(A) = \min(m,n) & \text{full-rank} \\ \text{rank}(A) < \min(m,n) & \text{rank-deficient} \end{cases}$$

رتبه کامل

رتبه ناقص

A is full-rank

A has full rank

$A \in \mathbb{R}^{n \times n}$

$\text{rank}(A) = n$ full-rank

$\text{rank}(A) < n$ rank-deficient

Linear Equations

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Linear Equations

3 equations معادله

$$\begin{cases} x+z=4 \\ x-y=3 \\ x+y+z=2 \end{cases}$$

$$1x+0y+1z=4$$
$$1x+(-1)y+0z=3$$
$$1x+1y+1z=2$$

3 unknowns x, y, z
مجهول

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$A \vec{x} = \vec{b}$$

m equations
n unknowns

Independent Equations

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Eq1 $2x + y + 2z = 6$

Eq2 $x - y = 3$

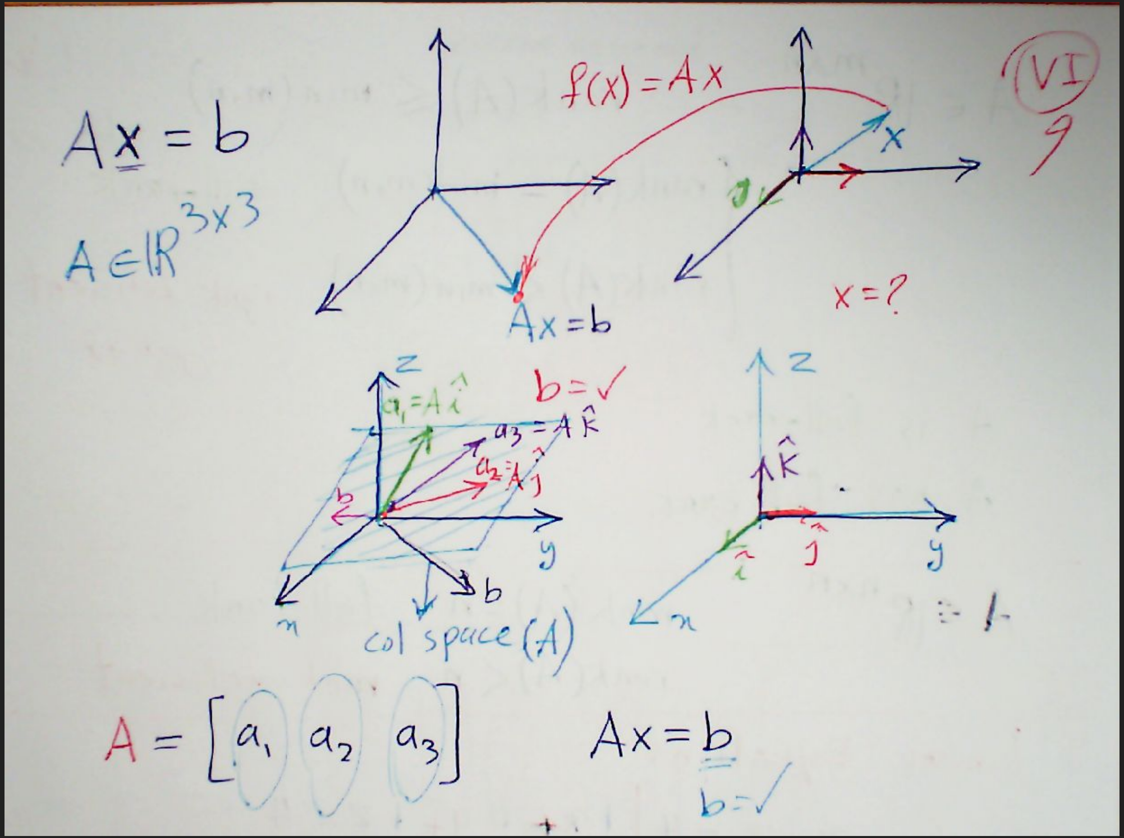
Eq3 $x + 2y + 2z = 3$

3 Equations

$$\underbrace{\begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}}_{\text{rank deficient}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

Eq1 − Eq2 = Eq3 2 Independet Equations

Geometric Interpretation



Let's focus on a special case

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$$\underbrace{A}_{\mathbb{R}^{m\times n}}\ \underbrace{x}_{\mathbb{R}^{n}} = \underbrace{b}_{\mathbb{R}^{m}}$$

m equation

n unknowns

$$\text{Special case} \begin{cases} m = n \Rightarrow A \text{ square} \left(\begin{array}{l} n \text{ Equations} \\ n \text{ Unknowns} \end{array}\right) \\ A \text{ has full rank } \left(\begin{array}{l} \text{Independent} \\ \text{equations} \end{array}\right) \end{cases}$$

$$\equiv \begin{cases} A \in \mathbb{R}^{n\times n} \quad (\cancel{\text{rank}}) \\ \text{rank}(A) = n \end{cases}$$