# Linear Algebra for Computer Science

## Lecture 9

## Solving Linear Equations, Inverse Matrix

# Linear Equations - Geometric Interpretation

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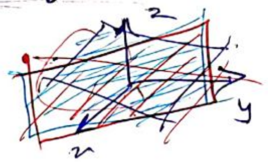
$$A \quad \vec{x} = b$$

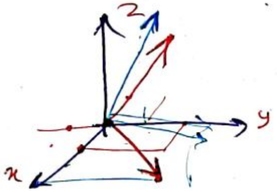
$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$2x + y + z = 3$

$x - y = 3$




$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} z = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$




$$\boxed{A\vec{x} = b}$$

# Review: Linear Equations: special case

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$$\underbrace{A}_{\mathbb{R}^{m\times n}}\ \underbrace{x}_{\mathbb{R}^{n}} = \underbrace{b}_{\mathbb{R}^{m}}$$

m equation

n unknowns

$$\text{Special case} \begin{cases} m = n \Rightarrow A \text{ square } \left( \begin{array}{l} n \text{ Equations} \\ n \text{ Unknowns} \end{array} \right) \\ A \text{ has full rank } \left( \begin{array}{l} \text{Independent} \\ \text{equations} \end{array} \right) \end{cases}$$

$$\equiv \begin{cases} A \in \mathbb{R}^{n\times n} \ \cancel{\text{rank}} \\ \operatorname{rank}(A) = n \end{cases}$$

# Singular and Nonsingular Matrices

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$A \in \mathbb{R}^{n \times n}$ square

$A$ has full rank, nonsingular

$\text{rank}(A) = n$, nondegenrate

Invertible معکوس پذیر

# Nonsingular Matrices and Linear Transformations

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$A \in \mathbb{R}^{n\times n}$ square

A <u>has full rank</u>, <u>nonsingular</u>

$\text{rank}(A) = n$, nondegenerate

Invertible معکوس پذیر

one-to-one
injective یک به یک

onto, surjective پوشا

$f(\vec{x}) = A\vec{x}$ . $A \in \mathbb{R}^{n\times n}$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

find $x$ s.t. $f(x) = b$

یک به یک

$\boxed{x=y} \Leftarrow x-y=0 \Leftarrow d=0$

$$\Uparrow$$

$$f(x) = f(y) \Rightarrow Ax = Ay \Rightarrow A\underbrace{(x-y)}_{d} = 0$$

$$Ad = 0 \Rightarrow \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = d_1 a_1 + d_2 a_2 + \cdots + d_n a_n = 0 \Rightarrow \begin{cases} d_1 = 0 \\ d_2 = 0 \\ \vdots \\ d_n = 0 \end{cases}$$

# Nonsingular Matrices and Linear Transformations



$f: \mathbb{R}^n \to \mathbb{R}^n$

$y \in \mathbb{R}^n \quad \exists x \overset{?}{} \quad f(x) = y$

$Ax \overset{?}{=} y$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = y$$

$a_1, \ldots, a_n$ form a basis for $\mathbb{R}^n \Rightarrow \exists\, x_1, x_2, \ldots, x_n$

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = y$$

$$\Rightarrow \exists x \in \mathbb{R}^n \Rightarrow Ax = f(x) = y$$

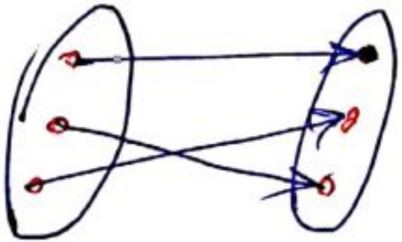
$f$ پوشا است

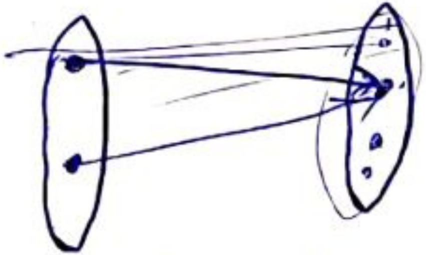
# Nonsingular Matrices and Linear Transformations



$f$ is one-to-one and onto $\Rightarrow$

$\exists f^{-1}$ $\quad f^{-1}(f(x)) = x$ for all $x \in \mathbb{R}^n$





$f$ is linear and bijective $\qquad f(x) = Ax$

$f^{-1}$ is ? and //

# Nonsingular Matrices and the Inverse Map



$f$ is linear and bijective $\qquad f(x) = Ax$

$f^{-1}$ is ? and //

$x_1 \doteq f^{-1}(y_1) \Rightarrow y_1 = A x_1$

$x_2 \doteq f^{-1}(y_2) \Rightarrow y_2 = A x_2$

$$f^{-1}(\alpha y_1 + \beta y_2)$$

$$= f^{-1}(\alpha A x_1 + \beta A x_2) = f^{-1}(A(\alpha x_1 + \beta x_2)) = f^{-1}(f(\alpha x_1 + \beta x_2))$$

$$= \alpha x_1 + \beta x_2 \Rightarrow \boxed{f^{-1}(\alpha y_1 + \beta y_2) = \alpha f^{-1}(y) + \beta f^{-1}(y)}$$

$f^{-1}$ is linear

# The Inverse Matrix

$\therefore f^{-1}$ is linear

$f^{-1}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is linear $\Rightarrow \exists B \Rightarrow f^{-1}(y) = By$

$\forall y \in \mathbb{R}^n$

$B = [b_1\ b_2 \cdots b_n]$ $\quad b_i = f^{-1}(e_i) = f^{-1}\left(\begin{array}{cc}\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} & \leftarrow i\end{array}\right)$

# The Inverse Matrix

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$$f^{-1}(f(x)) = x \Rightarrow B(Ax) = x \Rightarrow (BA)x = x \quad \forall x$$

$$\boxed{(AB)C = A(BC)}$$

$$\Rightarrow \cancel{(BA)} \quad (BA)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\boxed{\begin{aligned} IA &= A \\ AI &= A \end{aligned}}$$

$$(BA)\,I = I \Rightarrow BA = I$$

# The Inverse Matrix

$$(BA)\,I = I \Rightarrow BA = I$$

$f(x) = Ax$

$f^{-1}(x) = Bx$

B is called the inverse of A and is denoted by $A^{-1}$ $\Rightarrow$ Every nonsingular matrix has an Inverse

$\boxed{A^{-1}A = I}$ $\boxed{AA^{-1} = I}$ $\boxed{\forall x \quad AA^{-1}x = x}$

$$(A^{-1})^{-1} = A$$

$$\begin{cases} A^{-1}A = I \Rightarrow AA^{-1} = I \Rightarrow \overbrace{BA}^{I}A^{-1} = B \Rightarrow A^{-1} = B \\ BA = I \Rightarrow \end{cases}$$

# Practice: Examples of Inverse

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Rotation

Reflection

Shear

Scale

# Solve linear equations using Inverse Matrix

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$$Ax = y \Rightarrow \underline{\text{find } A^{-1}} \Rightarrow A^{-1}Ax = A^{-1}y$$

How?

$$\Rightarrow \boxed{x = A^{-1}y}$$

np.linalg.inv(A)

# Solve linear equations using Inverse Matrix

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$$Ax = y \Rightarrow x = A^{-1}y$$

$$\underset{n\times n}{\overset{\checkmark}{A}}\ \underset{n\times p}{\overset{?}{X}} = \underset{n\times p}{\overset{\checkmark}{Y}} \Rightarrow X = A^{-1}Y$$

$$A\begin{bmatrix} x_1 & x_2 & \cdots & x_p \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & \cdots & y_p \end{bmatrix}$$

# Null vectors of nonsingular matrices

$$\begin{bmatrix} & A & \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

~~A singular $\Rightarrow \exists\, \beta_1, \beta_2, \ldots, \beta_n$~~

A is nonsingular $\quad x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0$

$$\Rightarrow x_1 = x_2 \cdots = x_n = 0$$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

A is nonsingular $\quad \boxed{Ax = 0 \Rightarrow x = 0}$

# Null vectors of singular matrices

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A is singular $\exists\, x \neq 0$ s.t. $Ax = 0$

null vector

$\Rightarrow$ A has at least one nonzero null vetor.

# Singular Matrices and linear equations

$A \in \mathbb{R}^{n\times n}$ is singular = rank deficient

$Ax = b$

$rank(a) < n$

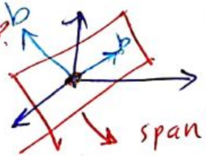
$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b \Rightarrow b = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

$b \in span(a_1, \ldots, a_n)$

$dim(span(a_1, \ldots, a_n)) < n$

column space

جواب ندارد $b$



$\mathbb{R}^n$

$span(a_1, a_2, \ldots, a_n)$

$colspace(A)$

$$A^{-1}Ax = \overset{x}{\cancel{I}} \quad \forall x$$

$$A^{-1}\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} x = \cancel{I}\, x$$

$$\cancel{A}\, A\underbrace{A^{-1}x}_{y \in C(A} = x \quad \forall x$$

$y \in C(A)$

column space

# Singular Matrices don't have an inverse

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A is nonsingular $\boxed{Ax = 0 \Rightarrow x = 0}$

A is singular $\exists\, x \neq 0$ s.t. $Ax = 0$

null vector

$\Rightarrow$ A has at least one nonzero null vetor.

A is singular $A^{-1}Ax = x$

Let $\boxed{x \neq 0}$ be a null vetor of A

نقض

$A^{-1}\underline{Ax} = x \Rightarrow A^{-1}0 = x \Rightarrow \boxed{x = 0}$

# Elimination

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$$
\begin{aligned}
\text{Eq1} &\quad \begin{cases} x + y = 7 \\ 2x - y = \cancel{1}5 \end{cases} \\
\text{Eq2}&
\end{aligned}
$$

$$\text{Eq2} - 2\text{Eq1} \Rightarrow 0 \cdot x - 3y = -9$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$$

$$x + y = 7 \Rightarrow x = 4$$

$$-3y = -9 \Rightarrow y = 3$$

# Intro to Elimination

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$$\begin{aligned} &\text{Eq1} \quad \begin{cases} x + y = 7 \\ 2x - y = 5 \end{cases} \\ &\text{Eq2} \end{aligned}$$

$$x + y = 7 \Rightarrow x = 4$$

$$-3y = -9 \Rightarrow y = 3$$

$$\text{Eq2} - 2\,\text{Eq1} \Rightarrow 0 \cdot x - 3y = -9$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$$

# Upper Triangular Matrices

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$$\overset{A}{\begin{bmatrix} 1 & 5 & 8 & 10 \\ 0 & 2 & 6 & 9 \\ 0 & 0 & 3 & 7 \\ 0 & 0 & 0 & 4 \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}} \overset{=}{=} \overset{b}{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}$$

VI

$4t = d \Rightarrow \boxed{t = \frac{t}{4}}$

$3z + 7t = c \Rightarrow \boxed{z = \checkmark}$

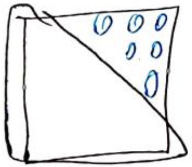
$2y + 6z + t = b \Rightarrow \boxed{y = \checkmark}$

$x + 5y + 8z + 10t \Rightarrow \boxed{x = \checkmark}$

بالا مثلثی

Upper Triangular

$Ax = b$



پایین مثلثی

Lower Triangular