Linear Algebra for Computer Science

Lecture 10

Elimination

Review: Upper Triangular Matrices

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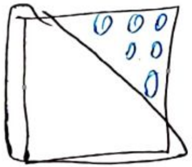
$$\overset{A}{\begin{bmatrix}1&5&8&10\\0&2&6&9\\0&0&3&7\\0&0&0&4\end{bmatrix}} \overset{X}{\begin{bmatrix}x\\y\\z\\t\end{bmatrix}} = \overset{b}{\begin{bmatrix}a\\b\\c\\d\end{bmatrix}}$$

بالا مثلثی

Upper Triangular

$Ax=b$



پایین مثلثی

Lower Triangular

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$4t = d \Rightarrow \boxed{t = \frac{t}{4}}$

$3z + 7t = c \Rightarrow \boxed{z = \checkmark}$

$2y + 6z + t = b \Rightarrow \boxed{y = \checkmark}$

$x + 5y + 8z + 10t \Rightarrow \boxed{x = \checkmark}$

Elimination

$a_1 \quad a_2 \quad a_3 \cdots \quad a_n$ are linearly independent

$\underbrace{a_1 + \alpha a_2}_{a_1'} \quad a_2 \quad a_3 \cdots \quad a_n$ are linearly independet

Elimination

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 11 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 3 & 1 & -1 & -2 \\ 2 & -3 & 1 & 11 \end{array}\right]$$

$$\begin{array}{c} \\ r_2 - = 3r_1 \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -10 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -20 \\ 11 \end{bmatrix}$$

Gaussian

Elimination

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\begin{bmatrix}1&2&3\\3&1&-1\\2&-3&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-2\\11\end{bmatrix} \qquad \left[\begin{array}{ccc|c}1&2&3&6\\3&1&-1&-2\\2&-3&1&11\end{array}\right] \quad 1$$

$$r_2 -= 3r_1 \quad \begin{bmatrix}1&2&3\\0&-5&-10\\2&-3&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-20\\11\end{bmatrix}$$

Gaussian Elimination

$$r_3 -= 2r_1 \quad \begin{bmatrix}1&2&3\\0&\boxed{-5}&\boxed{-10}\\0&\boxed{-7}&\boxed{-5}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-20\\-1\end{bmatrix}$$

$$r_3 -= \frac{7}{5}r_2 \quad \begin{bmatrix}1&2&3\\0&-5&-10\\0&0&9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-20\\27\end{bmatrix} \Rightarrow 9z=27 \Rightarrow z=3$$

$$x+\overset{-4}{2y}+\overset{+9}{3z}=6 \Rightarrow x=1$$

$$-5y-10\overset{3}{z}=-20 \Rightarrow y=-2$$

Back-Substitution

$$\left[\begin{array}{ccc|c}1&2&3&6\\ \end{array}\right.$$

Gauss-Jordan Elimination

$r_3 -= \frac{7}{5} r_2$ $\begin{bmatrix} 0 & -5 & -10 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} z \end{bmatrix} \begin{bmatrix} 27 \end{bmatrix} \Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & 0 & 9 & 27 \end{array}\right]$$

row echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$r_2 /= -5$

$r_3 /= 9$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$$

$r_1 -= 3r_3$

$r_2 -= 2r_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$$

Gauss-Jordan

$r_1 -= 2r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \Rightarrow \begin{array}{l} x = 1 \\ y = -2 \\ z = 3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

reduced row echelon form

Pivoting

$$\text{pivot} \leftarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 3 & 6 & 1 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 11 \end{bmatrix}$$

$$\begin{array}{l} \\ r_2 -= 3r_1 \\ r_3 -= 2r_1 \end{array} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 6 \\ 3 & 6 & 1 & -6 \\ 2 & -3 & 1 & 11 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & \boxed{0} & -8 & -24 \\ 0 & -7 & -5 & -1 \end{array}\right]$$

$$\text{switch}\begin{pmatrix} r_2 \\ r_3 \end{pmatrix} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & \boxed{-7} & -5 & -1 \\ 0 & 0 & -8 & -24 \end{bmatrix} \rightarrow \text{pivot}$$

Elimination as Matrix Multiplication

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$$\overset{E_1}{\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}}\left[\begin{array}{ccc|c}1&2&3&6\\3&1&-1&-2\\2&-3&1&11\end{array}\right]=\left[\begin{array}{ccc|c}1&2&3&6\\0&-5&-10&-20\\2&-3&1&11\end{array}\right]$$

$r_2=(-3)r_1+r_2$

$\cancel{A\ b}$

$\boxed{A\vec{x}=b}$

$\boxed{E_1A\vec{x}=E_1b}$

$$r_2=r_2-3r_1\quad\begin{bmatrix}1&2&3&6\\0&-5&-10&-20\\2&-3&1&11\end{bmatrix}$$

$$\overset{E_2}{\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}}\begin{bmatrix}1&2&3&6\\0&-5&-10&-20\\2&-3&1&11\end{bmatrix}=\left[\begin{array}{ccc|c}1&2&3&6\\0&-5&-10&-20\\0&-7&-5&-1\end{array}\right]$$

$r_3=(-2)r_1+r_3$

$\boxed{E_2E_1A\vec{x}=E_2E_1b}$

$$\overset{E_3}{\begin{bmatrix}1&0&0\\0&1&0\\0&-\frac{7}{5}&1\end{bmatrix}}\begin{bmatrix}1&2&3&6\\0&-5&-10&-20\\0&-7&-5&-1\end{bmatrix}=\left[\begin{array}{ccc|c}1&2&3&6\\0&-5&-10&-2\\0&0&9&27\end{array}\right]$$

$r_3=r_3-\frac{7}{5}r_2$

$\boxed{E_3E_2E_1A\vec{x}=E_3E_2E_1b}$

بالا مثلثی

E_1, E_2, E_3 پایین مثلثی

$\Rightarrow E_3E_2E_1$ پایین مثلثی

Pivoting and the Permutation matrix

$$\left[\begin{array}{ccc|c} 0 & -7 & -5 & -1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & -8 & -24 \\ 0 & -7 & -5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -5 & -1 \\ 0 & 0 & -8 & -24 \end{bmatrix}$$

An Example of permutation matrix

$$\underbrace{P_1 E_2 E_1}_{\text{بالا مثلثی}} A = P_1 E_2 E_1 b$$

Singular Matrices and Row Echelon Form

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$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 4 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -4 & -8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

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row echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

row echelon fom

$$\begin{bmatrix} X & \cdots & & & \\ 0 & Y & \cdots & & \\ 0 & 0 & Z & \cdots & \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$X \neq 0 \quad Y \neq 0 \quad Z \neq 0$

Solving Multiple Equations

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$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ -2 & -7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ -2 & -7 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ -2 & -7 & 4 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 4 & 3 \\ 5 & -2 \end{bmatrix}$$

$$A\vec{x}_1 = \vec{b}_1$$

$$A\vec{x}_2 = \vec{b}_2$$

$$A\vec{x}_3 = \vec{b}_3$$

$$A\ \vec{x}_4 = b_4$$

$$A\left[\vec{x}_1\ \vec{x}_2\ \vec{x}_3\ \vec{x}_4\right] = \left[\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4\right]$$

$$\underset{n\times n}{A}\ \underset{n\times p}{X} = \underset{n\times p}{B}$$

$$X = A^{-1}B$$

Solving Multiple Equations

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$$\begin{bmatrix}1&2&3\\3&2&1\\-2&-7&4\end{bmatrix}\begin{bmatrix}x&u\\y&v\\z&w\end{bmatrix}=\begin{bmatrix}-2&2\\4&3\\5&-2\end{bmatrix}$$

$$\underset{n\times n}{A}\ \underset{n\times p}{X} = \underset{n\times p}{B}$$

$$X = A^{-1}B$$

$$\left[\begin{array}{ccc|cc}1&2&3&-2&2\\3&2&1&4&3\\-2&-7&4&5&-2\end{array}\right]$$

$\Downarrow$ Gauss Jordan

$$\left[\begin{array}{ccc|cc}1&0&0&a&b\\0&1&0&c&d\\0&0&1&e&f\end{array}\right] \Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x&u\\y&v\\z&w\end{bmatrix}=\begin{bmatrix}a&b\\c&d\\e&f\end{bmatrix}$$

$$\begin{bmatrix}x&u\\y&v\\z&w\end{bmatrix}=\begin{bmatrix}a&b\\c&d\\e&f\end{bmatrix}$$

$$A\left[X_1\ X_2 \cdots X_p\right] = \left[b_1\ b_2 \cdots b_p\right]$$

$$\underset{n\times n}{A}\ \underset{n\times p}{X} = \underset{n\times p}{B}$$

$$AX = I$$

Solving Multiple Equations

$$A\begin{bmatrix} x_1 & x_2 & \cdots & x_p \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix}$$

$$\underset{n\times n}{A}\;\underset{n\times p}{X} = \underset{n\times p}{B}$$

$$\left[\begin{array}{c|c} A & B \end{array}\right] \qquad \left[\begin{array}{c|cccc} A & b_1 & b_2 & \cdots & b_p \end{array}\right]$$

$$\Big\downarrow \text{ Gauss Jordan}$$

$$\left[\begin{array}{c|c} I & C \end{array}\right] \qquad X = C$$

Find Inverse by Solving Multiple Equations

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$$AX = I$$

$$\begin{bmatrix} A & I \end{bmatrix}$$

$$\downarrow \text{Gauss Jordan}$$

$$\begin{bmatrix} I & F \end{bmatrix}$$

$$F = A^{-1}$$