Linear Algebra for Computer Science

Lecture 11

LU decomposition

Review: Solving multiple equations

$\underset{n\times n}{A} \quad \underset{n\times p}{B}$

$\underset{n\times n}{A}\ \underset{n\times p}{X} = \underset{n\times p}{B}$

I

$A\,[x_1\ x_2 \cdots x_p] = [b_1\ b_2 \cdots b_p]$

$\underset{n\times n}{A}$

$A x_1 = b_1 \qquad x_1 = \checkmark$

$A x_2 = b_2 \qquad x_2 = \checkmark$

$\cancel{\underset{n\times(n+p)}{[A\ X]} = [A\ x_1\ x_2 \cdots x_p]}$

~~↓ Gauss-Jordan~~

$\cancel{[I\ Y]}$

$\underset{n\times(n+p)}{[A\ B]} = [A\ b_1\ b_2 \cdots b_n]$

↓ Gauss-Jordan

$[I\ Y]$

$I X = Y \qquad \boxed{X = Y}$

$\underset{n\times n}{A X = I}$

Review: Find inverse matrix by elimination

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$n \times n$

$$AX = I$$

$$A \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[A \,\middle|\, I \right] = \left[A \;\; \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right]$$

$$\Big\downarrow \text{ Gauss Jordan}$$

$$\left[I \,\middle|\, Y \right] \Rightarrow IX = Y$$

$$X = Y = A^{-1}$$

Solve equations directly or using inverse matrix?

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$Ax = b$

1. $[A \;\; b] \xrightarrow{G/GJ} [I \;\; x]$ $\quad x = \checkmark$

   `np.linalg.solve(A,b)`

2. find A^{-1}

   $x = A^{-1}b$

   `Ainv = np.linalg.inv(A)`

   `x = Ainv@b`

$[A \;\; I] \xrightarrow{GJ} [I \;\; A^{-1}]$

Solve multiple equations directly or using inverse matrix?

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When should we use the Inverse matrix?

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$$A \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$$

$$A x_i = b_i$$

$A = \checkmark$

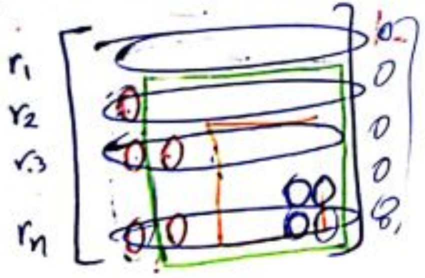
for $A^{-1} = \checkmark$

get b_i one-by-one

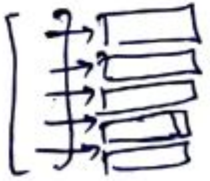
(eg. from camera
microphone
sensor)

$$x_i = A^{-1} b_i$$

Elimination: Computation Complexity



A



n operation

$$(n-1)n + (n-2)(n-1) + (n-3)(n-2) + \cdots + 1\times 2$$

$$O(n^3)$$

$$\sim \int_2^n (n-1)n$$

$$A^{-1} \to O(n^3)$$

Inverse of multiplication

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$$(AB)^{-1}$$

$$ABX = I$$

$$(AB)(B^{-1}A^{-1})$$

$$= A(BB^{-1})A^{-1}$$

$$= AIA^{-1}$$

$$= AA^{-1} = I$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Inverse of transpose

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$A \in \mathbb{R}^{m\times n}$ $\quad A^T \in \mathbb{R}^{n\times m}$ $\quad A^T_{ij} = A_{ji}$

$$\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}^T = \begin{bmatrix}1&3&5\\2&4&6\end{bmatrix}$$

$(AB)^T = B^T A^T$

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Transpose

$(A^T)^{-1}$

$$AA^{-1} = I \Rightarrow (AA^{-1})^T = I^T$$

$$\boxed{(A^{-1})^T} A^T = I$$

$$(A^{-1})^T = (A^T)^{-1} \overset{def}{=} A^{-T}$$

Inverse transpose

Inverse of diagonal matrices

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^{-1} = ?$$

$$\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Multiplying by diagonal matrices

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Inverse of Upper/Lower Triangular Matrices

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$$\begin{bmatrix} a & 0 & 0 \\ d & b & 0 \\ f & e & c \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ v & w & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1/a & & \\ & \cancel{b}\,1/b & \\ & & 1/c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ v & w & 1 \end{bmatrix}^{-1}$$

Inverse of Upper/Lower Triangular Matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$3 - 8 + x = 0$$

Matrix Decomposition

$$A = \underline{L} + \underline{V}$$

low-rank decomposition

$$A_{m\times n} = \underset{m\times r}{B}\ \underset{r\times n}{C}$$

$$\left[\quad\right] = \underset{n\times r}{\left[\quad\right]}\ \underset{r\times n}{\left[\quad\right]}$$

$\text{rank}(A) = r \leq \min(m, n)$

V

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Low-Rank matrix decomposition (factorization)

$$\Big[\; A \;\Big] = \underset{10^6\times 100}{\Big[\, B \,\Big]}\;\underset{100\times 10^6}{\big[\;\; C \;\;\big]}$$

A : $10^6 \times 10^6$

$\text{rank}(A) = 100$

10^{12} single precision

A: $4\times 10^{12} \Rightarrow 4000$ GB

4T

Ax : 10^{12} operations

$x \in \mathbb{R}^{1000000}$

B, C : 800 MB

$$Ax = (BC)x = B(Cx) = \underset{10^6\times 100}{B}\Big(\underbrace{\overset{10^8}{\overbrace{\underset{100\times 10^6}{C}\; x}}}_{y}\Big)$$

$$= \overset{10^8}{\overbrace{\underset{10^6\times 100}{B}\, y}} \qquad y\in\mathbb{R}^{100} \qquad \simeq 2\times 10^8$$

$10^{12}\times 10^{-9} = 1000$

$2\times 10^{8}\times 10^{-9} = 0.5$ s

LU decomposition - 2 x 2 matrices

LU decomposition

VI

$$\begin{bmatrix}1&2\\3&4\end{bmatrix} \xRightarrow{r_2 -= 3*r_1} \begin{bmatrix}1&2\\0&-2\end{bmatrix}$$

$$\begin{bmatrix}1&0\\-3&1\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}1&2\\0&-2\end{bmatrix}$$

$$E \quad A = U$$

$$\begin{bmatrix}1&2\\3&4\end{bmatrix} = E^{-1}\begin{bmatrix}1&2\\0&-2\end{bmatrix}$$

lower triangular

$$\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&0\\-3&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&2\\0&-2\end{bmatrix}$$

$$A = L \quad U$$

LU decomposition - Existence

$$\begin{bmatrix}0 & b\\ c & d\end{bmatrix} = \begin{bmatrix}x & 0\\ z & y\end{bmatrix}\begin{bmatrix}u & v\\ 0 & w\end{bmatrix} = \begin{bmatrix}xu & ?\\ ? & ?\end{bmatrix}$$

rank = 2

$$\Rightarrow xu = 0 \begin{cases} x = 0 \\ \text{or} \\ u = 0 \end{cases} \qquad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & b\\ c & d\end{bmatrix} = LU$$

$$PA = LU$$

LU decomposition

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{7}{5} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -10 \\ 0 & 0 & 9 \end{bmatrix}$$

$$E_{32} \qquad E_{31} \qquad E_{21} \qquad A \qquad U$$

$$A = (E_{32}E_{31}E_{21})^{-1}U = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$$

LU decomposition

$$\underset{E_{32}}{\begin{bmatrix}1&0&0\\0&1&0\\0&-\frac{7}{5}&1\end{bmatrix}}\underset{E_{31}}{\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}}\underset{E_{21}}{\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}}\underset{A}{\begin{bmatrix}1&2&3\\3&1&-1\\2&-3&1\end{bmatrix}}=\underset{U}{\begin{bmatrix}1&2&3\\0&-5&-10\\0&0&9\end{bmatrix}}$$

$$A=(E_{32}E_{31}E_{21})^{-1}U=E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$$

$$\begin{bmatrix}1&2&3\\3&1&-1\\2&-3&1\end{bmatrix}=\underbrace{\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&\frac{7}{5}&1\end{bmatrix}}\underbrace{\begin{bmatrix}1&2&3\\0&-5&-10\\0&0&9\end{bmatrix}}_{U}$$

$$=\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&\frac{7}{5}&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\\boxed{3}&1&0\\\boxed{2}&\boxed{\frac{7}{5}}&1\end{bmatrix}\begin{bmatrix}1&2&3\\0&-5&-10\\0&0&9\end{bmatrix}$$

$$A=LU$$

LU decomposition - Existence

$$A = LU$$

$$PA = LU$$

$$\begin{bmatrix} 0 & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} b & & \\ 0 & & \\ 0 & & \\ 0 & & \\ 0 & & \end{bmatrix}$$

Solving linear equations using LU decompotion

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$A\,x = b$

$A = L\,U$

$L\,U\,x = y$

- Let $z = U\,x$
- solve $L\,z = y$ for z