Linear Algebra for Computer Science

Lecture 13
Computing the null space, (reduced) row-echelon form

Review - null space, row echelon form

Null spae $A\underline{x} = 0$

$$N(A) = \{x \mid Ax = 0\}$$

$$A \in R^{m \times n} \quad N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

Row echelon Form

$$\begin{bmatrix} \boxed{12} & 13 & 2 & 4 & 8 \\ 0 & 0 & \boxed{7} & 8 & -2 \\ 0 & 0 & 0 & \boxed{7} & 4 \\ 0 & 0 & 0 & 0 & \boxed{8} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \boxed{12} & 13 & 2 & 4 \\ 0 & 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Review - null space, elimination, row echelon form

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$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 10 & 2 \\ 4 & 8 & 13 & 6 \end{bmatrix} \xRightarrow[r_4 - 4r_1]{r_2 - 2r_1,\ r_3 - 3r_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & -10 \\ 0 & 0 & 1 & -10 \end{bmatrix}$$

$$r_2 \Leftrightarrow r_3 \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & \boxed{1} & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -10 \end{bmatrix} \xRightarrow{r_4 - r_2} \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{1} & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $= 2$

$\dim(N(A)) = 2$

$$\dim(N(A)) = 2$$

How to represent a linear space?

How to represent null space? #2

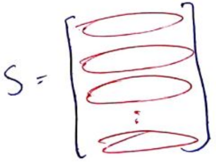
How to represent a linear ~~spa~~ subspace?

$A \in \mathbb{R}^{m\times n}$

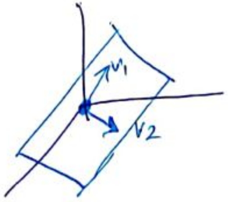
~~A linear subs~~

An r-dimentional linear subspace of $\boxed{\mathbb{R}^n}$ can be represented as a set of (r) basis vector.

$$S = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix}}_{n \times r}$$



representation is non-unique



Review: Elimination and Null space

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E invertible

$$Ax = 0 \iff EAx = 0$$

Elimination and Null space

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$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{1} & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \boxed{0} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} y \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & \boxed{2} & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Review: Elimination and Null space

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$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & \boxed{2} & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

invertible

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 2 & 4 & 5 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} y \\ t \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = -\begin{bmatrix} 2 & 4 & 5 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} y \\ t \\ u \end{bmatrix}$$

Elimination and Null space

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invertible

$$\begin{bmatrix}1 & 3\\0 & 2\end{bmatrix}\begin{bmatrix}x\\z\end{bmatrix} + \begin{bmatrix}2 & 4 & 5\\0 & 3 & -2\end{bmatrix}\begin{bmatrix}y\\t\\u\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 3\\0 & 2\end{bmatrix}\begin{bmatrix}x\\z\end{bmatrix} = -\begin{bmatrix}2 & 4 & 5\\0 & 3 & -2\end{bmatrix}\begin{bmatrix}y\\t\\u\end{bmatrix}$$

$$\begin{bmatrix}y\\t\\u\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 3\\0 & 2\end{bmatrix}\begin{bmatrix}x\\z\end{bmatrix} = \begin{bmatrix}-2\\0\end{bmatrix} \Rightarrow \begin{bmatrix}x\\z\end{bmatrix} = \begin{bmatrix}-2\\0\end{bmatrix} \qquad \begin{bmatrix}x\\y\\z\\t\\u\end{bmatrix} = \begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}y\\t\\u\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 3\\0 & 2\end{bmatrix}\begin{bmatrix}x\\z\end{bmatrix} = \begin{bmatrix}-4\\-3\end{bmatrix} \Rightarrow \begin{matrix}x = 1/2\\ z = -3/2\end{matrix} \qquad \begin{bmatrix}x\\y\\z\\t\\u\end{bmatrix} = \begin{bmatrix}1/2\\0\\-3/2\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}y\\t\\u\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 3\\0 & 2\end{bmatrix}\begin{bmatrix}x\\z\end{bmatrix} = \begin{bmatrix}-5\\2\end{bmatrix} \Rightarrow \begin{matrix}x = -8\\ z = 1\end{matrix} \qquad \begin{bmatrix}x\\y\\z\\t\\u\end{bmatrix} = \begin{bmatrix}-8\\0\\1\\0\\1\end{bmatrix}$$

$$N = \begin{bmatrix}-2 & 1/2 & -8\\1 & 0 & 0\\0 & -3/2 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

(a basis for $N(A)$)

$$\underset{m\times n}{A}\ \underset{n\times(n-r)}{N} = 0_{m\times(n-r)}$$

Reduced row-echelon form

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$$\begin{bmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & \boxed{2} & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xRightarrow{r_2/=2} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3/2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

All pivots $= 1$

$$\xRightarrow{r_1 = r_1 - 3r_2} \begin{bmatrix} \boxed{1} & 2 & 0 & -1/2 & 8 \\ 0 & 0 & \boxed{1} & 3/2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

→ reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 0 & -1/2 & 8 \\ 0 & 0 & 1 & 3/2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 2 & -1/2 & 8 \\ 0 & 3/2 & -1 \end{bmatrix} \begin{bmatrix} y \\ t \\ u \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} -2 & +1/2 & -8 \\ 0 & -3/2 & +1 \end{bmatrix} \begin{bmatrix} y \\ t \\ u \end{bmatrix}$$

$$\begin{array}{c} x \\ y \\ z \\ t \\ u \end{array} \begin{bmatrix} ? & ? & ? \\ 1 & 0 & 0 \\ ? & ? & ? \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1/2 & -8 \\ 1 & 0 & 0 \\ 0 & -3/2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reduced row-echelon form

$A \in \mathbb{R}^{m\times n}$

VI

$$Ax = 0 \xRightarrow[\text{reduced row\_echelon form}]{\text{Gauss-Jordan}} \begin{array}{cc} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{c} r \\ m-r \end{array} & \begin{bmatrix} I & G \\ 0\;0 & 0\;0\;0\;0 \\ 0\;0 & 0\;0\;0\;0 \\ 0\;0 & 0\;0\;0\;0 \end{bmatrix} \end{array} \quad \overbrace{Px}^{y} = 0$$

B

permutation matrix

$r = \text{rank}(A)$

$$[I\ G]\,y = 0 \Rightarrow [I\ G]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow y_1 = -G\,y_2$$

$$\begin{bmatrix} -G \\ I_{(n-r)} \end{bmatrix}$$

$$\begin{bmatrix} -G \end{bmatrix} = -G \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N(B) = \begin{bmatrix} -G \\ I \end{bmatrix}$$

$$A = BP$$

$Ax = 0$

$B\underline{Px} = 0 \Rightarrow$

$By = 0 \qquad Px = y$

$x = P^T y = P^{-1} y$

General Ax=b

$$Ax=b \qquad A\in\mathbb{R}^{m\times n} \qquad x\in\mathbb{R}^n \qquad b\in\mathbb{R}^m$$

VII

$$\text{rank}(A)=r\leq\min(m,n)$$

$$N(A)=\checkmark$$

$$\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=b \Rightarrow x_1\vec{a}_1+x_2\vec{a}_2+\cdots+x_n\vec{a}_n=b$$

$$b\in\text{span}(a_1,a_2,\dots,a_n)=C(A)$$

$b\notin C(A)$ $\quad$ $Ax=b$ has no solutions ~~answers~~

$$b\in C(A)\Rightarrow \exists\, x_1,x_2,\dots,x_n \text{ s.t.}$$

$$x_1\vec{a}_1+x_2\vec{a}_2+\cdots+x_n\vec{a}_n=\vec{b}$$

$$A\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=b\Rightarrow$$ A has at least one solution!

$$\exists\vec{x}:\quad A\underline{\vec{x}}=\vec{b} \qquad \vec{n}\in N(A)\quad A\vec{n}=0$$