Linear Algebra for Computer Science

Lecture 14

General Linear Equations

Solutions to General Linear Equations

$\underline{\underline{Ax=b}}$

$\cancel{A: m\times n}$ $A\in\mathbb{R}^{m\times n}$

$\cancel{b\in}$ $b\in\mathbb{R}^{m}$

$x\in\mathbb{R}^{n}$

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$b\notin C(A)$ No solution ?

$b\in C(A)$ At least one solution (x_p)

$A x_p = b$

$A x_n = 0 \quad x_n\in N(A)$

$A(x_p+x_n) = Ax_p + Ax_n = b+0=b$

$A(x_p + \underbrace{\alpha x_n}_{\in N(A)}) = b$

$$\begin{cases} Ax_p = b \\ Ax = b \end{cases} \Rightarrow A(x-x_p)=0 \Rightarrow \overbrace{x-x_p}^{x_n}\in N(A)$$

$x = x_p + x_n \quad x_n\in N(A)$

Solutions to General Linear Equations

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$Ax = b$

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$b \notin C(A)$ No Solution!

$b \notin C(A)$ at least one solution!

1. Find x_p s.t. $Ax_p = 0$
2. Find $N(A)$ (a basis for $N(A)$)

all solutions to $Ax=b$ are

$$\{ x_p + x_n \mid x_n \in N(A) \}$$

a) How to check if $b \in C(A)$?

b) How to find x_p?

Set of solutions is not a linear subspace

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$\boxed{A: m \times n}$

$$S = \{\underline{x} \mid Ax = b\} \subseteq \mathbb{R}^n$$

Does S form a linear subspace of $\mathbb{R}^n$?

$b = 0 \Rightarrow$ YES $\quad x = N(A)$

$b \neq 0 \quad Ax = b \quad A(\alpha x) = \alpha b \neq b \Rightarrow \alpha x \notin S$

$\boxed{\text{let } \alpha \neq 1}$

$x, y \in S \quad \left.\begin{array}{l} Ax = b \\ Ay = b \end{array}\right\} \quad A(x+y) = Ax + Ay = 2b \neq b \quad x + y \notin S$

$x, y \in S \quad A(\alpha x + \beta y) = \underline{(\alpha + \beta) b \neq b}$ in general

S is not a linear subspace

Set of solutions is not a linear subspace

The set of solutions to $Ax=b$ III

for $b \neq 0$ is not a linear subspace.

$x, y \in S = \{x \mid Ax = b\}$

$$A(\alpha x + \beta y) = (\alpha + \beta) b$$

what linear combinations of x and y are in S?

$\alpha x + \beta y \in S$ if and only if $\alpha + \beta = 1$

$\Longrightarrow$ $\alpha x + \beta y$ is an affine combination

Affine subspace

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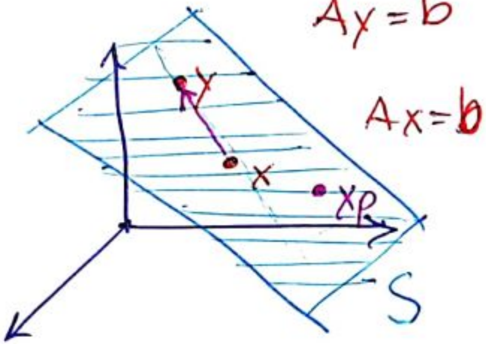
$\Longrightarrow \alpha x + \beta y$ is an affine combination

$y - x \in N(A)$



$N(A) = \{x - x_p \mid Ax = b\}$

is a linear subspace

$D(S) = D(N(A))$

S is an affine subspace (of $\mathbb{R}^n$)

Full column-rank case

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$A \in \mathbb{R}^{m \times n}$ A has full column rank IV

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$\text{rank}(A) = r = n$

($\Leftrightarrow$ A has full ~~col~~-rank)

~~$\Rightarrow m \geq n$~~ $r = n \Rightarrow m \geq n \Rightarrow A = \begin{bmatrix} | & | & | & | \\ | & | & | & | \end{bmatrix}$

$\Rightarrow$ A is a tall matrix

(thin, skinny)

$r = n \Rightarrow$ A has n independent columns

$Ax = b$

$N(A)$ $Ax = 0 \Rightarrow \begin{bmatrix} a_1 & a_2 \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$

$\Rightarrow x = 0$

$\Rightarrow N(A) = \{0\}$

$b \notin C(A) \Rightarrow$ No Solution

$b \in C(A) \Rightarrow$ A unique solution!

Full row-rank case

$Ax = b$ $\quad A \in \mathbb{R}^{m \times n}$ has full row rank $\quad$ V 15

$$r = \text{rank}(A) = m$$

$\Rightarrow m \leq n$ $\quad$ ($\Rightarrow$ A has full rank

$$A = \begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \end{bmatrix}$$

A is a fat matrix
wide

$Ax = b$ $\quad \text{rank}(A) = m$

$\Rightarrow$ A has $\underline{m}$ idependent column

$$C(A) = \mathbb{R}^m$$

$\Rightarrow$ b is always in $C(A)$

$\Rightarrow$ there is Always a solution.

$$\dim(N(A)) = n - r = n - m$$

$$x = x_p + x_n \qquad x_n \in N(A)$$

Non-singular case

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$A \in \mathbb{R}^{n \times n}$ is non-singular

$$\left.\begin{cases} A \text{ has full column rank} \Rightarrow \text{No or unique solution} \\ A \text{ has full row rank} \Rightarrow \text{has a solution} \end{cases}\right\} \Rightarrow$$

$\Rightarrow$ has a unique solution!

Rank-deficient case

$Ax = b \qquad A \in \mathbb{R}^{m \times n}$

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A is rank-deficient $\qquad r < \min(m, n)$

n

$b \notin C(A)$ No Solution

$b \in C(A)$

$\dim(N(A)) = n - r \geq 1$

$b \in C(A) \Rightarrow$ infinite solutions

Find all solutions by elimination

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$$Ax = b$$

1- check if $b \in C(A)$

2- find a particular solution

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 9 & -6 \\ & & & & \\ & & & & \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \\ -1 \end{bmatrix} \quad b$$

$$Ax = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 3 & 6 & 10 & 2 & 4 \\ 4 & 8 & 13 & 6 & 9 \end{bmatrix} x = \begin{bmatrix} 12 \\ 24 \\ 19 \\ 31 \end{bmatrix} = b \qquad 31 \rightarrow \boxed{32}$$

$Ax = b$

$EAx = Eb$

$\Rightarrow b \in C(A)$

$\Updownarrow$

$Eb \in C($

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -10 & -11 & -17 \\ 0 & 0 & 1 & -10 & -11 & -17 \end{array}\right] \quad -17 \rightarrow \boxed{-16}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 12 \\ 0 & 0 & 1 & -10 & -11 & -17 \\ 0 & 0 & 1 & -10 & -11 & -17 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad -17 \rightarrow -16$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 12 \\ 0 & 0 & 1 & -10 & -11 & -17 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad 0 \rightarrow 1$$

$b \in C(A)$ ✓

Find all solutions by elimination

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$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 12 \\ 0 & 0 & 1 & -10 & -11 & -17 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Downarrow \text{GJ}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 34 & 38 & 97 \\ 0 & 0 & 1 & -10 & -11 & -17 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 2 & 0 & 34 & 38 \\ 0 & 0 & 1 & -10 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix} = \begin{bmatrix} 97 \\ -17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 2 & 34 & 38 \\ 0 & -10 & -11 \end{bmatrix} \begin{bmatrix} y \\ t \\ u \end{bmatrix} = \begin{bmatrix} 97 \\ -17 \end{bmatrix}$$

$$\vec{X}_p = \begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix} = \begin{bmatrix} 97 \\ 0 \\ -17 \\ 0 \\ 0 \end{bmatrix}$$

$$N(A) = C\left(\begin{bmatrix} -2 & -34 & -38 \\ 1 & 0 & 0 \\ 0 & 10 & 11 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

Find all solutions by elimination

solutions $\vec{x}_p + \vec{x}_n$ $\quad x_n \in N(A)$

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$$\text{solutions} = \begin{bmatrix} 97 \\ 0 \\ -17 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & -34 & -38 \\ 1 & 0 & 0 \\ 0 & 10 & 11 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 97 \\ 0 \\ -17 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 & -34 & -38 \\ 1 & 0 & 0 \\ 0 & 10 & 11 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha = -1 \\ \beta = 3 \\ \gamma = -1 \end{bmatrix}$$