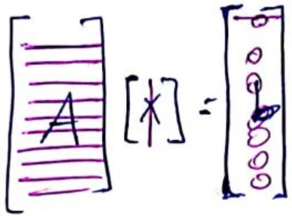
Linear Algebra for Computer Science

Lecture 16

Projection into a linear subspace

Overdetermined equations and noise

$Ax = b$



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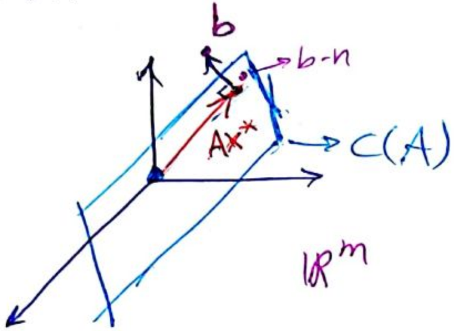
$$Ax = \vec{b} - \vec{n} \qquad b - n \in C(A)$$

$A: m \times n$



$$b - Ax^* \perp Ax^*$$

$$b - Ax^* \perp Ax \quad \forall x$$

$$x^* = \underset{x}{\operatorname{argmin}} \|Ax - b\|$$

Orthogonality

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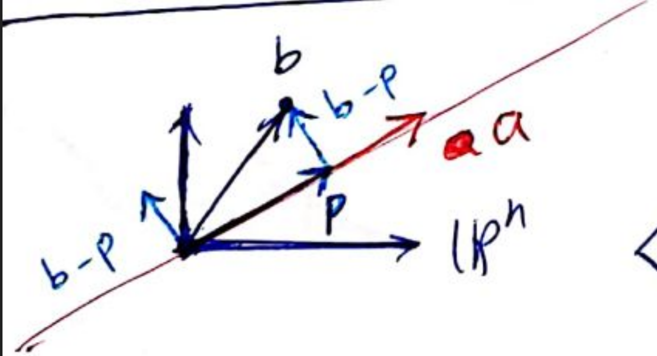
$$\langle u, v \rangle = u^T v = v^T u = \sum_{i=1}^{n} u_i \cdot v_i$$

$u, v \in \mathbb{R}^n$

$$\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{\sum u_i^2}$$

$$u \perp v \iff u^T v = 0$$

Projection into a linear space - 1D case



$\vec{p} = x\,\vec{a}$ $\quad x \in \mathbb{R}$ scalar $\quad \boxed{[\vec{a}]\,x = [b]}$

$\langle a, b-p \rangle = 0 \Rightarrow a^T(b-p) = 0$

$a^Tb - a^Tp = 0 \Rightarrow a^Tb = a^Tp = x\,a^Ta \Rightarrow x = \dfrac{a^Tb}{a^Ta} = \dfrac{a^Tb}{\|a\|^2} \in \mathbb{R}$

$p = x\vec{a} = \left(\dfrac{a^Tb}{a^Ta}\right) a$

Matrix multlipicaltion of a vector and a scalar

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$\alpha \vec{V}$ → vector

$\alpha \in \mathbb{R} \quad V \in \mathbb{R}^n$

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$$\underset{1\times 1}{\alpha}\ \underset{n\times 1}{\vec{V}} = \underset{n\times 1}{\vec{V}}\ \underset{1\times 1}{\alpha}$$

→ compatible with matrix multiplication

$\alpha = u^T V \in \mathbb{R}$

$\alpha = u^T A v \ \mathbb{R}$

$(u^T v) V \neq u^T v V$

cannot multiply

$$= V (u^T v) = \underbrace{\underbrace{\underset{n\times 1}{V}\ \underset{1\times n}{u^T}}_{n\times n}\ \underbrace{\underset{n\times 1}{V}}_{n\times 1}}_{n\times 1}$$

$$\alpha V = (u^T V) v = v (u^T v) = (v u^T) v$$

$$= v (v^T u) = v v^T u$$

$$= \underbrace{(v v^T)}_{\substack{n\times n \\ \text{Rank}=1}} u$$

$\vec{b}$ $\vec{a}$

Projection is a linear operation

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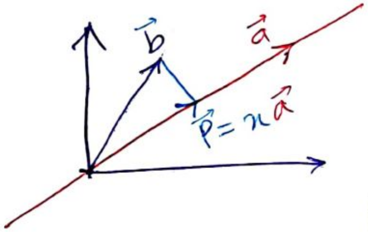


$n \times n$

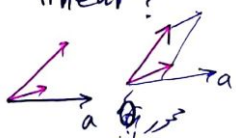
Rank=1

$$x = \frac{a^Tb}{a^Ta} = \frac{a^Tb}{\|a\|^2}$$

$$p = x\,a = \left(\frac{a^Tb}{a^Ta}\right)a$$

$p = f(b)$ f: projection on subspace spanned by $\vec{a}$

is f linear? $f(\alpha b) \overset{?}{=} \alpha f(b)$ ✓



$f(ab+c) = f(a) + f(c)$ ✓

f is linear

$\boxed{\text{projection is a linear operation}}$

$p = f(b) \underset{\text{خطی}}{=} Ab \quad A \in \mathbb{R}^{n\times n}$

$$f(b) = p = x\,a = \underbrace{\left(\frac{a^Tb}{a^Ta}\right)a}_{f(b)}$$

The projection matrix

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$$f(b) = p = x\,a = \left(\frac{a^Tb}{a^Ta}\right)a = \underset{n\times 1}{a}\;\underset{1\times 1}{\left(\frac{a^Tb}{a^Ta}\right)} = \frac{1}{a^Ta}\,a(a^Tb)$$

compatible
with matrix
product

$$[\;][\;\;\;]$$
$$\underset{n\times 1}{\;}\quad\underset{1\times n}{\;}$$

$$= \frac{1}{a^Ta}\left(\underset{n\times 1}{a}\,\underset{1\times n}{a^T}\right)b = \Bigg(\underbrace{\frac{1}{a^Ta}\,\underbrace{aa^T}_{n\times n}}_{A = P_a}\Bigg)\,b = \left(\frac{\overset{n\times n}{aa^T}}{\underset{1\times 1}{a^Ta}}\right)b$$

$$f(b) = \boxed{P_a}\,b$$

projection
matrix

$\mathbb{R}^3$

The projection operation

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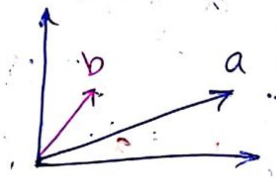


projection matrix

$$P_a = \frac{aa^T}{a^Ta} = \frac{aa^T}{\|a\|^2}$$

$$f_a(b) = P_a b = \frac{aa^T}{a^Ta} b$$

$$f_{2a}(b) = f_{ka}(b) = f_a(b)$$

$$f_a(2b) = 2f_a(b)$$

$$\vec{b} = \alpha \vec{a} \Rightarrow f_a(b) = P_a b = b$$

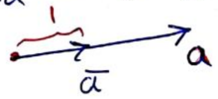
$$P_a = \frac{aa^T}{a^Ta} = \frac{aa^T}{\|a\|^2} = \underbrace{\left(\frac{a}{\|a\|}\right)}\overbrace{\left(\frac{a}{\|a\|}\right)}^{\bar{a}}{}^T$$

$$\bar{a} = \frac{a}{\|a\|} = \frac{a}{\sqrt{a^Ta}}$$

$$\|\bar{a}\| = 1$$

$\bar{a}$ is a unit vector

$$P_a = \bar{a}\,\bar{a}^T$$



Properties of the projection matrix

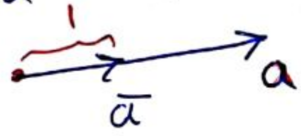
$$P_a = \frac{aa^T}{a^Ta} = \frac{aa^T}{\|a\|^2} = \left(\frac{a}{\|a\|}\right)\overbrace{\left(\frac{a}{\|a\|}\right)}^{\bar{a}}{}^T$$

$$\bar{a} = \frac{a}{\|a\|} = \frac{a}{\sqrt{a^Ta}}$$

$\|\bar{a}\| = 1$

$\bar{a}$ is a unit vector

$$P_a = \bar{a}\,\bar{a}^T$$



$$P_a^T = \frac{aa^T}{\|a\|^2} = \frac{1}{\|a\|^2}(aa^T)^T = \frac{aa^T}{\|a\|^2} = P_a$$ Symmetric

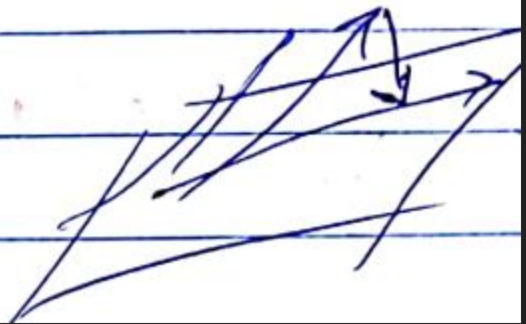
$$\cancel{P_a P_a x} \quad P_a \underbrace{P_a b}_{=\alpha\vec{a}} = P_a b \quad \forall b \Rightarrow P_a P_a = P_a$$

Properties of the Projection Matrix

$f_a(b) = f_a(f_a(b))$ idempotent

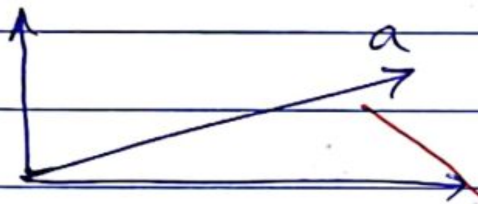
$P_a = P_a P_a$

projection matrix $\begin{cases} PP = P \\ P^T = P \end{cases}$



Rank of the 1-D projection matrix

$$C(P_a) = C\left(\frac{a a^T}{\|a\|^2}\right) = C(a) = \{\alpha \vec{a} \mid \alpha \in R\}$$



$$\frac{a a^T}{\|a\|^2} = \frac{\begin{bmatrix} a \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}}{\|a\|^2}$$

$$\begin{bmatrix} \frac{a_1}{\|a\|^2} a & \frac{a_2}{\|a\|^2} a & \cdots & \beta a \end{bmatrix}$$

$$\mathrm{Rank}(P) = 1 = \dim(S)$$

Length of the projected vector

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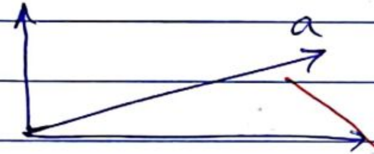
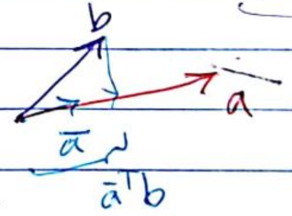


$$\frac{a a^T}{\|a\|^2} = \frac{\begin{bmatrix} a \end{bmatrix} [a_1\ a_2 \cdots a_n]}{\|a\|^2}$$

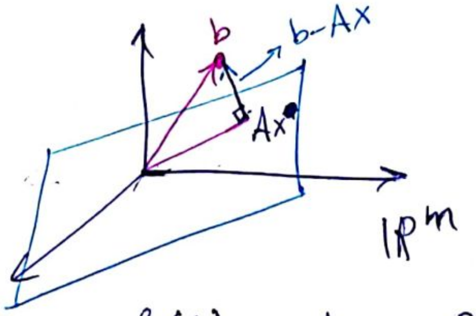
$$\begin{bmatrix} \frac{a_1}{\|a\|^2} a & \frac{a_2}{\|a\|^2} a & \cdots & a \end{bmatrix}$$

$$\mathrm{Rank}(P) = 1 = \dim(S)$$

$$f_a(b) = (\bar{a}\,\bar{a}^T)\, b = \bar{a}\,(\bar{a}^T b) = \bar{a}\,\langle \bar{a}, b \rangle$$



General Projection into a linear subspace



$A \in \mathbb{R}^{m\times n}$

$Ax = b$

A has full column rank
(independent columns)

$A = [a_1 \; a_2 \cdots a_n]$

$C(A) = \mathrm{span}(a_1 — a_n)$

$= \{Ax \mid x \in \mathbb{R}^n\}$

$\subseteq \mathbb{R}^m$

$f_A(b) = Ax = P_A b$?

projection map
(is linear)

$(b - Ax) \perp C(A)$

$(b - Ax) \perp y$ for all $y \in C(A)$
for all $y = Az = [a_1 \; a_2 \cdots a_n]\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

General Projection into a linear subspace

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$(b-Ax) \perp y$ for all $y \in C(A)$

for all $y = Az = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

$= z_1 a_1 + z_2 a_2 + \dots + z_n a_n$

$\Rightarrow \langle b - Ax, Az \rangle = 0 \quad \forall z \in \mathbb{R}^n$

$(Az)^T (b - Ax) = 0 \quad \forall z$

$z^T A^T (b - Ax) = 0 \quad \forall z \in \mathbb{R}^n$

$z^T = \begin{matrix} 1\,0\,0\,0 \cdots \\ 0\,1\,0\,0 \end{matrix}$

$\underset{I}{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}} A^T (b - Ax) = 0$

$\boxed{\begin{bmatrix} A \end{bmatrix} [x] = \begin{bmatrix} b \end{bmatrix}}$

$\Rightarrow A^T (b - Ax) = 0 \Rightarrow A^T b = A^T A x$

$\underbrace{(A^T A)}_{n \times n} \underset{n\times 1}{x} = \underbrace{A^T b}_{n \times 1}$

$\underset{n\times m}{A^T} \underset{m\times n}{A} \underset{n\times 1}{x} = \underset{n\times m}{A^T} \underset{m\times 1}{b}$

$\begin{bmatrix} A^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} [x] = \begin{bmatrix} A^T \end{bmatrix} \begin{bmatrix} b \end{bmatrix}$

$\Rightarrow x = (A^T A)^{-1} A^T b$

Least squares solution

is $A^T A$ invertible?

$[\quad][x] = [\quad]$