Linear Algebra for Computer Science

Lecture 17

Projections

Projection and Least Squares Solution

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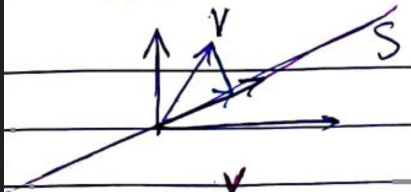
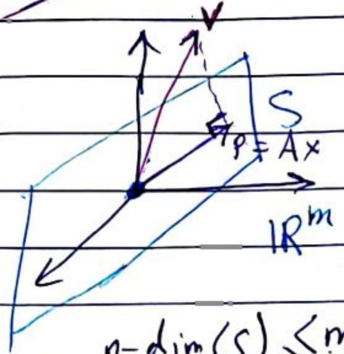


$f_S(V)$



$f_S(V) = Ax$ $\qquad x = \underset{x'}{\arg\min} \|Ax' - v\|$

$n = \dim(S) < m \qquad A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \Rightarrow S = C(A)$

$\mathrm{rank}(A) = n$

$a_1, a_2, \ldots, a_n$ a basis for S

$x = ?$

$p = f_S(v) = ?$

$$\boxed{\underbrace{(A^TA)}_{n\times n} x = \underbrace{A^T v}_{n\times 1}}$$

Projection and Least Squares Solution

$$x = (A^TA)^{-1}A^Tv$$

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} y \end{bmatrix} \in \mathbb{R}^m$$

A has full column rank

$$\begin{bmatrix} A^T \end{bmatrix}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix}$$

$$\exists x \quad A^TAx = 0 \implies \left.\begin{matrix} Ax \in C(A) \\ Ax \in N(A^T) \end{matrix}\right\} \xrightarrow[C(A)\perp N(A^T)]{} x = 0$$

$$\Downarrow$$

$$x^TA^TAx = x^T0 = 0 \implies (\underbrace{Ax}_{y})^T(\underbrace{Ax}_{y}) = y^Ty = 0 \implies \|y\| = 0$$

$$\implies y = 0$$

$$\implies Ax = 0 \implies x = 0$$

$$\implies \boxed{A^TA \text{ is non-singular}}$$

General Projection Matrix

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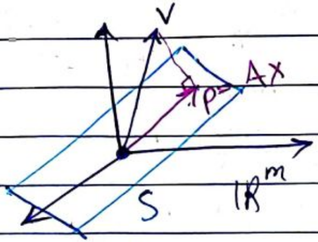


$$x = (A^TA)^{-1}A^Tv$$

least squares solution

$$p = f_S(v) = f_A(v) = Ax = \underbrace{A(A^TA)^{-1}A^T}_{\text{projection matrix}}v$$

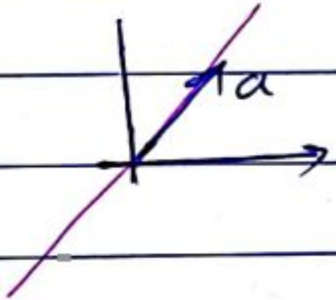
A: $m \times n$

$$\underset{}{f_A(v)} = \underset{m\times m}{P_A}\ \underset{m\times 1}{v} \in \mathbb{R}^m \qquad P_A = \underset{m\times n}{A}\ \underset{n\times n}{\underbrace{(A^TA)^{-1}}}\ \underset{n\times m}{A^T}$$

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} \begin{bmatrix} (A^TA)^{-1} \end{bmatrix} \begin{bmatrix} \quad n \times m \quad \end{bmatrix}$$

$$\begin{bmatrix} \\ P_A \\ \\ \end{bmatrix}$$

The Special 1D Case



$$\underset{m\times 1}{A} = \begin{bmatrix} a \end{bmatrix}$$

$$P_A = \underset{m\times 1}{\begin{bmatrix} a \end{bmatrix}} \underset{1\times 1}{\left[(a^Ta)^{-1}\right]} \underset{1\times m}{\begin{bmatrix} a^T \end{bmatrix}}$$

$$= \frac{a\,a^T}{(a^Ta)}$$

Properties of the Projection Matrix

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$$(a^Ta)$$

$$(P_A)^T = \left(A\,(A^TA)^{-1}A^T\right)^T = A^{T^T}\,(A^TA)^{-T}\,A^T$$

$$= A\,\left((A^TA)^T\right)^{-1}A^T$$

$$= A\,(A^TA)^{-1}A^T = P_A$$ symmetric

Properties of the Projection Matrix

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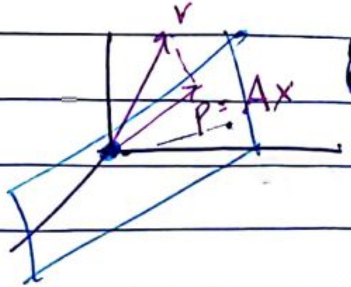
$$P_A v = A\underbrace{(A^TA)^{-1}A^Tv}_{y\in\mathbb{R}^n}$$

$$= Ay \in C(A) = S$$



$$P_A P_A v = P_A v \qquad f_A(f_A(v)) = f_A(v)$$

idempotent

$$P_A P_A = A(A^TA)^{-1}\underbrace{A^T A(A^TA)^{-1}}_{I}A^T = A(A^TA)^{-1}A^T = P_A$$

Properties of the Projection Matrix

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$$P_A = P_A^{\dagger}$$

$$P_A P_A = P_A$$

Rank and Column Space of the Projection Matrix

$$P_A(A) = A\,\underbrace{(A A)^{T} A^{T}}_{B}$$

$$C(P_A) \subseteq C(A)$$

$$C(A) \subseteq C(P_A)$$

$$\Rightarrow C(P_A) = C(A)$$

$$\mathrm{rank}(P_A) = n$$

Dimensions of the Projection Matrix

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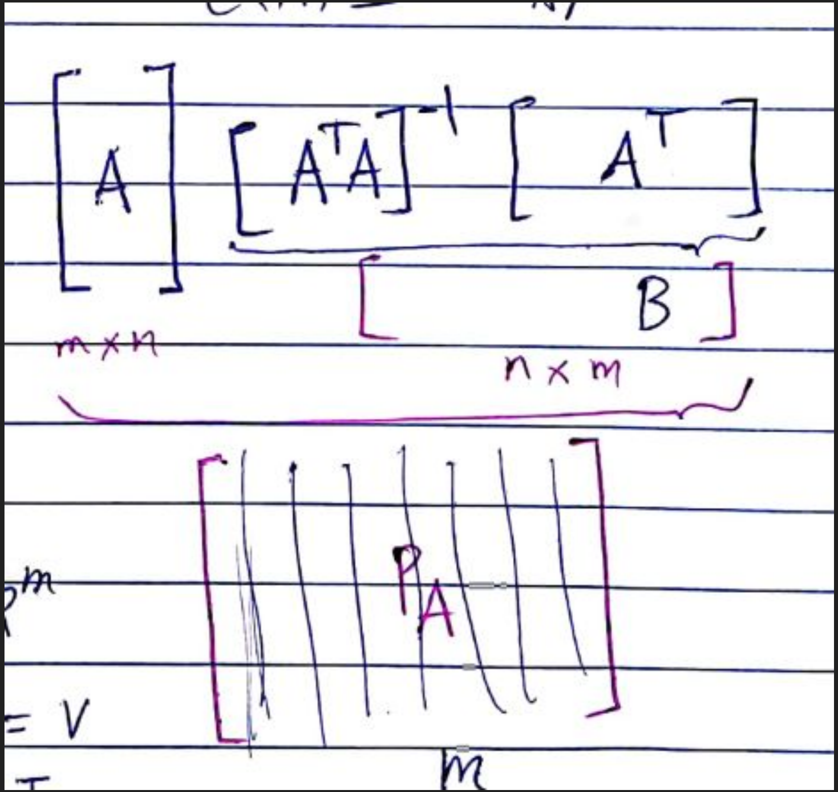

Projection into the embedding space R^m

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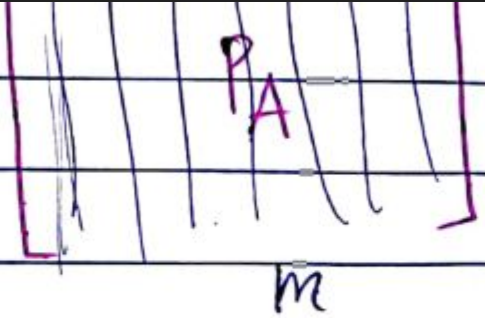
$\forall\, v \in \mathbb{R}^m$



$S = \mathbb{R}^m \Rightarrow P_S(v) = v$

$\Rightarrow P_A = I$

$A \in R^{m \times m}$

A: invertible

$$P_A = A(A^T A)^{-1} A^T = A\, A^{-1} A^{-T} A^T = I \cdot I = I$$

The orthonormal case

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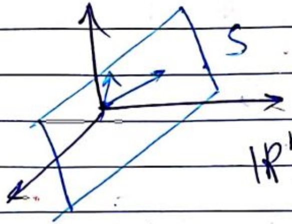

$\mathbb{R}^m$.

Let $a_1\ a_2 \cdots a_n$ are a basis for S.

$a_1, a_2, \cdots, a_n$ form an orthogonal basis for S

if $a_i \perp a_j$ $(a_i^T a_j = 0)$ for all $i \neq j$

$a_1, a_2, \cdots, a_n$ form an orthonormal basis for S

$$\text{if} \begin{cases} a_i^T a_j = 0 & i \neq j \\ \|a_i\| = 1 & (a_i^T a_i = 1) \end{cases}$$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} A \end{bmatrix} \end{array} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$a_1 \cdots a_n$ are orthonormal

The orthonormal case

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$$\underset{m}{A} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$a_1 \cdots a_n$ are orthonormal

$$\underset{n \times n}{A^TA} = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_n$$

$$AA^T = ?$$

A has orthonormal columns

$$P_A = A\underbrace{(A^TA)^{-1}}_{I}A^T = AA^T$$

Orthogonal Matrix

A has orthonormal columns $(A^T A = I)$

& A is square

$\Rightarrow A \in \mathbb{R}^{n\times n}$ has orthonormal column

$A^T A = I \Rightarrow A^T = A^{-1} \Rightarrow A A^T = I$

$\Downarrow$ A is an orthogonal matirx

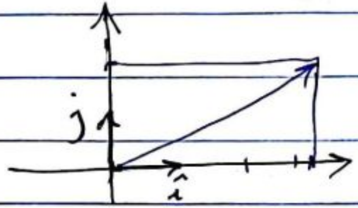
$\Downarrow$ A has orthonormal rows

$A = \begin{bmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{bmatrix}$ is orthogonal

Change of basis

Change of Basis

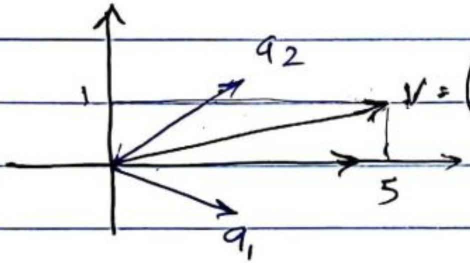
V



$$x=\begin{bmatrix}3\\2\end{bmatrix}=3\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}$$

$$=3\hat{i}+2\hat{j}$$

$$=3e_1+2e_2$$



$v=\begin{pmatrix}5\\1\end{pmatrix}$

what is the coordinate of v in basis $\{a_1, a_2\}$

Change of basis

$$V = \begin{pmatrix} 5 \\ 1 \end{pmatrix} = 5 \times \hat{i} + 1 \times \hat{j}$$

$$V = ?\, a_1 + ?\, a_2 = x_1\, a_1 + x_2\, a_2$$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ coordinates of V in basis (a_1, a_2)

write everything ~~on~~ $(\underline{V}, \underline{a_1}, \underline{a_2})$ in standard basis

$$V = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad a_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 2.5 \\ 1.5 \end{bmatrix}$$

$$x_1 a_1 + x_2 a_2 = V = \begin{bmatrix} a_1 & a_2 \end{bmatrix} x = V$$

solve linear equations

$$A x = V$$

Change of basis

find coordinates of ~~vector~~ $v \in \mathbb{R}^n$ in basis

$a_1, a_2, \ldots, a_n$

$$A: \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = v$$

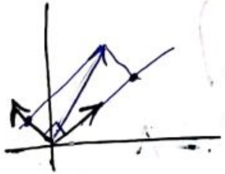
$$Ax = v$$

$$x = A^{-1} v$$

Change of basis

what if $a_1, a_2, \dots, a_n$ are orthonormal $\quad a_i^T a_i = 1$

$$A^{-1} = A^T \qquad a_i^T a_j = 0 \quad i \neq j$$



$$V = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = a_i^T V = a_i^T \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix}$$

$$\underbrace{\begin{array}{cccccccc} & & & i & & & & \\ 0 & 0 & \cdots 0 & 1 & 0 & 0 & 0 & \end{array}}_{e_i^T} \begin{bmatrix} x_1 \\ | \\ x_n \end{bmatrix} = x_i$$

$$\boxed{x_i = a_i^T V}$$

Orthogonal Subspaces

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$S_1, S_2 \subseteq V$ are linear subspaces of V.

S_1, S_2 are orthogonal if for all $v_1 \in S_1$, $v_2 \in S_2$ $v_1 \perp v_2$

R^3