Lab Instructions - session 5

Least Squares

Least Squares

Look at the following code.

least_squares.py

```
import numpy as np

x_true = np.array([3, 1.5,-1.0, 2.4, -3, -.1, 2.2, 4.1, -3.2, 1.0])
n = x_true.size # no. of unknowns

m = 20 # no of equations (measurements)
A = np.random.randn(m,n)

# create the measurments
y_true = A @ x_true

# add noise to the measurments
sigma = 0.01
measurement_noise = sigma * np.random.randn(m)
y_noisy = y_true + measurement_noise

# we have access to the matrix "A" and noisy measurements "y_noisy".
# Frome these, we intend to estimate "x_true" using least squares
x_est = np.linalg.inv(A.T@A) @ A.T @ y_noisy
# x_est = np.linalg.solve(A.T@A, A.T @ y_noisy)
# x_est = np.linalg.lstsq(A,y_noisy)[0]

# measure the distance between the estimated unkowns "x_est"
# and the ture ones "x_true"
print('error=', np.linalg.norm(x_est - x_true))
```

- Explain the code.
- Use the alternative method `np.linalg.solve(A.T@A, A.T @ y_noisy)` and check if you get a similar `x_est`. Why this is equivalent to the least squares solution `np.linalg.inv(A.T@A) @ A.T @ y_noisy`?
- You can also use the numpy function `np.linalg.lstsq` to do least squares. Verify that it gives the same result.

Task 1

Put the above in a loop to repeat it 100 times and report the average error. Afterward, keep increasing **`m`**, the number of equations (measurements). How does increasing the number of equations affect the average error? How do you explain this?

Back to the Face Models

From the previous lab, remember trying to find the **`a`**, **`b`**, and **`c`** to reconstruct **`TargetFace2`** as a linear combination of **`Face1`**, **`Face2`**, and **`Face3`**. To do that, we first created an overdetermined system of 136 equations in 3 unknowns **`F`** **x** = **t**, where **`F`** and **t** were obtained by

```
face1 = Face1.ravel()
face2 = Face2.ravel()
face3 = Face3.ravel()
t = TargetFace.ravel();
F = np.stack((face1, face2, face3), axis=1)
```

In the previous Lab session, we chose 3 out of 136 equations, randomly or otherwise, to find **x = [a,b,c]**^{T} as the solution to a system of 3 equations and 3 unknowns. You observed that this approach failed when the target face was noisy.

```
NoisyTargetFace = TargetFace + 3 * np.random.randn(*TargetFace2.shape)
```

Here, we intend to use all the 136 equations to solve for **x = [a,b,c]**^{T}.

Task 2

Use the least squares method to solve **`F`** **x** = **t** for a noisy target **t**. Compare the solution against when randomly selecting 3 points.

`task2.py`

```
import matplotlib.pyplot as plt
import numpy as np

from face_data import Face1, Face2, Face3, TargetFace2, edges

def plot_face(plt,X,edges,color='b'):
    "plots a face"

    plt.plot(X[:,0], X[:,1], 'o', color=color, markersize=3)

```

```
    for i,j in edges:
        xi = X[i,0]
        yi = X[i,1]
        xj = X[j,0]
        yj = X[j,1]

        # draw a line between X[i] and X[j]
        plt.plot((xi,xj), (yi,yj), '-', color=color)
    plt.axis('square')
    plt.xlim(-100,100)
    plt.ylim(-100,100)

TargetFace = TargetFace2
NoisyTargetFace = TargetFace + 3 * np.random.randn(*TargetFace.shape)

face1 = Face1.ravel()
face2 = Face2.ravel()
face3 = Face3.ravel()
t = NoisyTargetFace.ravel();

F = np.stack((face1, face2, face3), axis=1)

for i in range(5):
    inds = np.random.choice(range(136), 3, replace=False)

    a1,b1,c1 = # solve 3 random equations
    a2,b2,c2 = # least squares solution

    Face_rnd = a1 * Face1 + b1 * Face2 + c1 * Face3
    Face_lsq = a2 * Face1 + b2 * Face2 + c2 * Face3

    plot_face(plt, NoisyTargetFace, edges, color='k')
    plot_face(plt, Face_rnd, edges, color='g')
    plot_face(plt, Face_lsq, edges, color='b')

    plt.show()
```

- What do you conclude by comparing **`Face_rnd`** with **`Face_lsq`**?
- Plot **`Face_lsq`** against **`TargetFace`** instead of **`NoisyTargetFace`**. What do you observe?
- Which one do you think is closer to **`TargetFace`**? **`Face_lsq`** or **`NoisyTargetFace`**? Notice that we constructed **`Face_lsq`** from the noisy target **`NoisyTargetFace`**. Why do you think this happens?
- Confirm the above numerically, by computing the sum of squared differences between the elements of paris of matrices.
- Use **`numpy.linalg.lstsq`** to solve the least squares problem.