Mathematics for AI

Lecture 10
Eigenvalues and Eigenvectors, Algebraic Multiplicity, Eigenspaces, Geometric Multiplicity

Computing Eigenvalues

K. N. Toosi
University of Technology

Eig2 (I)

$A \in \mathbb{R}^{n \times n}$

$A\underline{v} = \underline{\lambda}\,\underline{v} \qquad v \in \mathbb{R}^n, \; \lambda \in \mathbb{R}$

$$Av = \lambda v \Rightarrow Av - \lambda v = 0 \Rightarrow \underset{n\times n}{A}v - \underset{n\times n}{(\lambda I)}v = 0$$

$$\Rightarrow \overbrace{(A - \lambda I)}^{n\times n}v = 0 \Longrightarrow (A - \lambda I) \text{ is singular}$$

$v \neq 0$

$\det(A - \lambda I) = 0 \Longrightarrow$ a polynomial on λ of degree n

چندجمله‌ای درجه n

Computing Eigenvalues

$$Av = \lambda v \Rightarrow Av = (\lambda I)v \Rightarrow Av - \lambda I v = 0$$

$(A - \lambda I)v = 0 \Rightarrow (A - \lambda I)$ has a non-zero null vector $\Rightarrow |A - \lambda I| = 0$

$$\left(A - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) v = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} v = 0$$

Example

$$A = \begin{bmatrix}0&3\\3&0\end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix}0&3\\3&0\end{bmatrix} - \begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix} = \begin{bmatrix}\lambda&3\\3&\lambda\end{bmatrix}$$

charcteristic
چندجمله‌ای مشخصه

$$\det\begin{bmatrix}\lambda&3\\3&\lambda\end{bmatrix} = 0 \Rightarrow \lambda^2 - 3^2 = 0 \Rightarrow (\lambda-3)(\lambda+3) = 0$$

چندجمله‌ای درجه ۲

$$\lambda = 3 \Rightarrow \begin{bmatrix}0&3\\3&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = 3\begin{bmatrix}v_1\\v_2\end{bmatrix} \Rightarrow \begin{bmatrix}3v_2\\3v_1\end{bmatrix} = \begin{bmatrix}3v_1\\3v_2\end{bmatrix} \Rightarrow v_1 = v_2$$

$$v = \alpha\begin{bmatrix}1\\1\end{bmatrix}$$

$$\lambda = -3 \Rightarrow \begin{bmatrix}0&3\\3&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = -3\begin{bmatrix}v_1\\v_2\end{bmatrix} \Rightarrow \begin{bmatrix}3v_2\\3v_1\end{bmatrix} = \begin{bmatrix}-3v_1\\-3v_2\end{bmatrix} \Rightarrow v_1 = -v_2$$

$$v = \alpha\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\left(\begin{bmatrix}1\\1\end{bmatrix}, 3\right), \left(\begin{bmatrix}-1\\1\end{bmatrix}, -3\right)$$

choose eigenvector to be unit vectors

$$\left(\begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\end{bmatrix}, 3\right), \left(\begin{bmatrix}-\sqrt{2}/2\\\sqrt{2}/2\end{bmatrix}, 3\right)$$

Example

eig2 II

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix}$$

$$\underbrace{(2-\lambda)^2 - 3^2}_{\text{charcteristic polynomial of } A} = 0 \Rightarrow 2-\lambda = \pm 3 \Rightarrow \begin{cases} \lambda = -1 \\ \lambda = 5 \end{cases}$$

$$\lambda = -1 \Rightarrow A - \lambda I \Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 5 \Rightarrow A - \lambda I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example

K. N. Toosi
University of Technology

Shear $A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & \alpha \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$$

ریشه مضاعف

repeated

$$(A-\lambda I)v = 0 \Rightarrow \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} v = 0 \Rightarrow \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Rightarrow \alpha v_2 = 0 \Rightarrow \begin{cases} \alpha = 0 \end{cases}$$

$\alpha v_2 = 0 \Rightarrow \begin{cases} \alpha = 0 & \text{(A is identity) every } v \neq 0 \in \mathbb{R}^2 \text{ is an eigenvector} \\ \alpha \neq 0 & \Rightarrow v_2 = 0 \Rightarrow v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} \end{cases}$

$\Rightarrow$ the only eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Algebraic and Geometric Multiplicity

K. N. Toosi University of Technology

$\alpha \neq 0$ MA 10

$$A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \qquad |A - \lambda I| = \begin{vmatrix} 1-\lambda & \alpha \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 = 0 \Rightarrow \lambda = 1$$

algebraic multiplicity

$$\begin{bmatrix} 1-1 & \alpha \\ 0 & 1-1 \end{bmatrix} \begin{bmatrix} V \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ 0 \end{bmatrix}$$

$\alpha \neq 0$

rank 1

$$\dim(N(A - \lambda I)) = 2 - 1 = 1$$

geometric multiplicity

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example

Rotation (2D) $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\theta = \pi/2 \Rightarrow R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$|R - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1$ no real roots

what about complex roots? $\lambda = \pm i$

$\lambda = i \Rightarrow (R - \lambda I)v = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda = -i \Rightarrow (R - \lambda I)v = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ i \end{bmatrix}$

Example

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos \pi/2 & -\sin \pi/2 \\ \sin \pi/2 & \cos \pi/2 \end{bmatrix}$$

eig2 III

$\Rightarrow$ no real eigenpairs $\Rightarrow$ there are complex eigenpairs

$$\left(\begin{bmatrix} 1 \\ i \end{bmatrix}, -i \right), \left(\begin{bmatrix} i \\ 1 \end{bmatrix}, +i \right)$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -i \\ 0 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Number of Eigenvalues

$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b & c & d \\ e & f-\lambda & g & h \\ i & j & k-\lambda & l \\ m & n & o & p-\lambda \end{bmatrix}$$

$\Rightarrow \det(A - \lambda I)$ is a polynomial of degree 4 on λ

$A \in \mathbb{R}^{n \times n} \Rightarrow \det(A - \lambda I)$ is a polynomial of ~~deg~~ degree n on λ

$\Rightarrow$ has at most n roots

$\Rightarrow A \in \mathbb{R}^{n\times n} \Rightarrow A$ has at most n eigenvalues

$A \in \mathbb{C}^{n\times n} \Rightarrow$ " " " " n "

Eigenspace

K. N. Toosi
University of Technology

Let λ be an eigenvalue of $A \in \mathbb{R}^{n \times n}$. eig2 IV

What are the set of ~~eig~~ eigenvectors correspondig to λ? $\Rightarrow N(A - \lambda I)$ $\Rightarrow$ is a linear subspace

null space

Eigenspace

v_1, v_2 are eigenvectors corresponding to eigenvalue λ.

v_1, v_2 share a common eigenvalue λ

$$\left.\begin{aligned} Av_1 &= \lambda v_1 \\ Av_2 &= \lambda v_2 \end{aligned}\right\} \quad A(\alpha v_1 + \beta v_2) = \alpha A v_1 + \beta A v_2$$

$$= \alpha \lambda v_1 + \beta \lambda v_2$$

$$= \lambda(\alpha v_1 + \beta v_2)$$

$$\Rightarrow \overbrace{\alpha v_1 + \beta v_2}^{\text{(every linear combination of } v_1, v_2)} \text{ is also an eigenvector of } A.$$

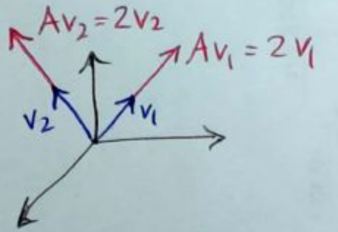
$\Rightarrow$ the set of eigenvetors corresponding to an eigenvalue λ is a linear subspace.

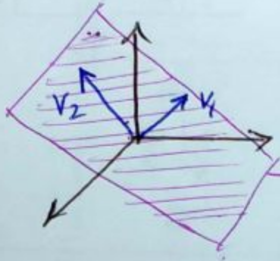
Eigenspace

In general for every eigenvalue λ there is a linear subspace V_λ such that for all $v \in V_\lambda$ we have $Av = \lambda v$

V_λ is called the eigenspace corresponding to λ.

$$V_\lambda = N(A - \lambda I)$$





eigenspace corresponding to $\lambda = 2$

Example: Identity matrix

eig2 V

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ the only eigenvalue is $\lambda = 1$

$\text{eigenspace}(\lambda = 1) = V_\lambda = \mathbb{R}^2 \Rightarrow$ 2D eigenspace

$Av = v = 1 \cdot v$

Identity matrix, geometric and algebraic multiplicity

$$A = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix}1-\lambda & 0\\ 0 & 1-\lambda\end{vmatrix} = (1-\lambda)^2 = 0$$

$$\lambda = 1$$

Algebraic mult = 2

$$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_x\\ v_y\end{bmatrix} = 0$$

eigenspace $(\lambda = 1) = \mathbb{R}^2$

geo mult. = 2

Example:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow \begin{array}{l} \lambda = 2 \Rightarrow V_\lambda = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\} \quad 1D \\ \lambda = 3 \Rightarrow V_\lambda = \left\{ \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\} \quad 1D \end{array}$$

Example:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\lambda = 2 \Rightarrow V_\lambda = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}$$ 1D

$$\lambda = 3 \Rightarrow V_\lambda = \left\{ \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$ 2D

Example:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow |A - \lambda I| = \begin{vmatrix} 2-\lambda & & \emptyset \\ & 3-\lambda & \\ \emptyset & & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)^2 = 0$$

$\lambda_1 = 2 \rightarrow$ alg mult $= 1$ ~~age~~ e. $\rightarrow \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2, \lambda_3 = 3 \rightarrow$ alg mult $= 2 \rightarrow \begin{bmatrix} 0 \\ \alpha \\ \beta \end{bmatrix}$

Basis for eigenspaces

for every eigenspace we can choose a basis 2D

$\lambda = 2$ basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ eigenbasis $\lambda = 2$

$\lambda = 3$ basis $\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ eigenbasis for $\lambda = 3$

np.linalg.eig

Complex Numbers

eig 3 I

$A \in \mathbb{R}^{m \times n} \longrightarrow A \in \mathbb{C}^{m \times n}$

$u \in \mathbb{R}^{n} \longrightarrow u \in \mathbb{C}^{n}$

$c = a + bi \qquad c \in \mathbb{C}$

$a, b \in \mathbb{R}$

$$u = \begin{bmatrix} 4 \\ 3i-2 \\ 6i \end{bmatrix} \in \mathbb{C}^3$$

$$\Downarrow$$

$$\bar{u} = \begin{bmatrix} 4 \\ -3i-2 \\ -6i \end{bmatrix}$$

$$\bar{c} = \overline{a + bi} = a - bi$$

conjugate مزدوج

$$A = \begin{bmatrix} 2 & 3+4i \\ i & 4 \\ 2-i & 1+i \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} 2 & 3-4i \\ -i & 4 \\ 2+i & 1-i \end{bmatrix}$$

Complex Numbers

$$\text{بردار}\begin{cases} \text{اندازه} & \|v\| \\ \text{تعامد} & v \perp u \end{cases}$$

$$u, v \in \mathbb{R} \begin{cases} \|v\|^2 = v^T v = \langle v, v \rangle \\ u \perp v \Rightarrow \langle u, v \rangle = 0 \Rightarrow u^T v = 0 \end{cases}$$

for complex vectors u, v $\quad \langle u, v \rangle = \bar{v}^T u$

$$\|u\|^2 = \bar{u}^T u = \langle u, u \rangle$$

$$u \perp v \iff \langle u, v \rangle = 0 \iff \boxed{\bar{v}^T u = 0}$$

Orthogonal Matrix => Unitary Matrix

K. N. Toosi
University of Technology

$U \in \mathbb{C}^{n\times n}$ is an orthogonal matrix

$$\Rightarrow U = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \text{ has orthonormal columns (rows)}$$

$$\left.\begin{array}{l} u_i \perp u_j \quad i \neq j \\ \|u_i\| = 1 \end{array}\right\} \Rightarrow \langle u_i, u_j \rangle = \bar{u}_j^T u_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\begin{bmatrix} \text{---} & \bar{u}_1^T & \text{---} \\ \text{---} & \bar{u}_2^T & \text{---} \\ & \vdots & \\ \text{---} & \bar{u}_n^T & \text{---} \end{bmatrix} \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = I$$

$$\boxed{\bar{U}^T U = I}$$

Conjugate Transpose (Hermitian Transpose)

$$(\bar{U})^T = \bar{U}^T = U^* = U^H$$

$$A \in \mathbb{C}^{m\times n} \qquad (\bar{A})^T = \bar{A}^T = A^* = A^H = A'$$

conjugate transpose of A

Hermitian transpose

MATLAB