Mathematics for AI

Lecture 11
Complex Matrices, Diagonalization, Eigendecomposition of Symmetric and Hermitian Matrices, Positive Definite Matrices

Real Symmetric Matrices

eigs IV

$A \in \mathbb{R}^{n \times n}$, symmetric $\Rightarrow$ Eigenvalues are real

$\Downarrow$ $\qquad$ $\Downarrow$

$\bar{A} = A$ $\qquad$ $A^T = A$

Eigenvectors are orthogonal

$\Downarrow$

(can be chosen to be) orthogonal

$\Downarrow$

if $\text{Dim}(\text{eigenspace}) \geq 2$

Complex dot product

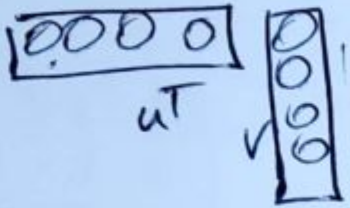
$u, v \in \mathbb{R}^n \Rightarrow \langle u, v \rangle = u^T v = v^T u$

$u, v \in \mathbb{C}^n \Rightarrow \langle u, v \rangle = u^T \bar{v} = \bar{v}^T u = v^* u = v^H u$

$\langle u, u \rangle = \bar{u}^T u = \cancel{\sqrt{}}\sum |u_i|^2$

$\bar{u}_i \cdot u_i$

$\overset{1\times 1}{u^T v} = \overset{1\times 1}{(u^T v)^T} = v^T u$



$= v^T u$

$\langle u, v \rangle = \overline{\langle v, u \rangle}$

$UU^T = U^T U = I \;/\; UU^H = U^H U = I$

Orthogonal matrices => Unitary Matrices

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$$UU^T = U^T U = I \quad / \quad UU^H = U^H U = I$$

Real Symmetric Matrices

$$Av = \lambda v \Rightarrow \underbrace{\bar{v}^T A v}_{\in \mathbb{R}} = \lambda \underbrace{\bar{v}^T v}_{\substack{\in \mathbb{R} \\ \neq 0}} \Rightarrow \lambda \in \mathbb{R}$$

$$\left.\begin{array}{l} A v_1 = \lambda_1 v_1 \\ A v_2 = \lambda_2 v_2 \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \Rightarrow \begin{array}{l} \bar{v}_2^T A v_1 = \lambda_1 \bar{v}_2^T v_1 \\ \bar{v}_1^T A v_2 = \lambda_2 \bar{v}_1^T v_2 \xrightarrow{\overline{(\;)}^T} \bar{v}_2^T A v_1 = \underset{\lambda_2}{\overset{\shortparallel}{\bar{\lambda}_2}}\, \bar{v}_2^T v_1 \end{array}$$

$$\Rightarrow \begin{array}{l} \bar{v}_2^T A v_1 = \lambda_1 \bar{v}_2^T v_1 \\ \bar{v}_2^T A v_1 = \lambda_2 \bar{v}_2^T v_1 \end{array}$$

$$\left.\begin{array}{l} \Rightarrow \lambda_1 \bar{v}_2^T v_1 = \lambda_2 \bar{v}_2^T v_1 \\ \lambda_1 \neq \lambda_2 \end{array}\right\} \Rightarrow \begin{array}{l} \bar{v}_2^T v_1 = 0 = \langle v_1, v_2 \rangle \\ \Rightarrow v_1 \perp v_2 \end{array}$$

Real Symmetric Matrices

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$A \in \mathbb{R}^{n\times n}$ is a real symetric matrix

all the eigenvalues are real.

the eigenvectors are (can be chosen to be) orthogonal.

(Also true for Hermitian matrices $\bar{A}^T = A$)

Independent Eigenvectors and Diagonaliza

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Assume that the matrix $A \in \mathbb{R}^{n\times n}$ has $\underline{\underline{n}}$ independent eigenvetors

(A has an eigenbasis)

Let $v_1, v_2, \dots, v_n$ be independent eigenvectors of $A \in \mathbb{R}^{n\times n}$ and $\lambda_1, \lambda_2 \cdots \lambda_n$ be the corresponding eigenvalues

$$Av_1 = \lambda_1 v_1 \;,\; Av_2 = \lambda_2 v_2 \;,\; \cdots \;,\; Av_n = \lambda_n v_n$$

$$\Rightarrow A\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} Av_1 & Av_2 & \cdots & Av_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix}$$

$$\Rightarrow A\underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{V}\underbrace{\begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ & & \ddots \\ \emptyset & & & \lambda_n \end{bmatrix}}_{\Lambda}$$

$$AV = V\Lambda \xrightarrow{v_1, v_2, \dots v_n \text{ independent}} A = V\Lambda V^{-1}$$

V: matrix of eigenvectors

Λ: diagonal matrix of eigen values

$A = V\Lambda V^{-1}$: ~~eigen decomp~~ eigendecomposition of A

Independent Eigenvectors and Diagonalization

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A has n independent ~~eigenval~~ eigenvectors

$\Longrightarrow$ A diagonalizable

$$A = V \Lambda V^{-1}$$

diagonal

Eigen-decomposition

A diagonalizable $\Rightarrow A = V \underbrace{D}_{\text{diagonal}} V^{-1} \Rightarrow AV = VD$

$$\Rightarrow A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

$A v_i = d_i \cdot v_i$ $\qquad$ $v_1, v_2, \ldots, v_n$ independent

$\Rightarrow A$ has n independent eigenvalues.

Eigen-decomposition - Symmetric Matrices

$A \in \mathbb{R}^{n \times n}$ is symmetric & has $\underline{n}$ independent eigenvectors

$$A = V \Lambda V^{-1} \qquad \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

A symmetric $\quad v_i \perp v_j \quad i \neq j \quad v_i^T v_j = 0$

choose eigen values to be unit vector

$\Rightarrow$ V is an orthogonal matrix $\Rightarrow V^{-1} = V^T$

$$\Rightarrow A = V \Lambda V^T \rightarrow$$ eigen-decomposition for symmetric matrices

Transformation in Eigenbasis

$$Ax = \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x \text{ in the eigenbasis}}}_{\text{transformed } x \text{ in eigenbasis}}}_{\text{back to the standard basis}}$$

transformed x = Ax

back to the standard basis

Ax in the standard basis

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$A = V \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} V^{-1}$$

Transformation in Eigenbasis

$f: \mathbb{C}^n \to \mathbb{C}^n$ linear $\quad x \mapsto Ax \quad f(x) = Ax$

& A is diagonalizable $\quad f(x), f(f(x)) = \cancel{f^2(x)}$

$f(f(f(f(\cdots))))$

$$y = f(x) = Ax = V\Lambda V^{-1}x$$

$$\boxed{U = V^{-1}} = U^{-1}\Lambda U x$$

$$\cdot\ y = Ax = U^{-1}\Lambda Ux \Rightarrow \underbrace{Uy}_{y'} = \Lambda \underbrace{Ux}_{x'} \Rightarrow y' = \Lambda x'$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ \emptyset & & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1' \\ \lambda_2 x_2' \\ \lambda_3 x_3' \end{bmatrix}$$

$$f(f(x)) = AAx = V\Lambda V^{-1}V\Lambda V^{-1}x = V\Lambda\Lambda V^{-1}x$$

$$= V\Lambda^2 V^{-1}x$$

$$\underbrace{f(f(f(x)))}_{\text{n}} = f^n(x) = A^n x = V\Lambda^n V^{-1}x.$$

Joint diagonalization

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Let A, B have the same eigen basis $\rightarrow$ A & B are jointly diagonalizable

$A \rightarrow (v_1, \lambda_1), (v_2, \lambda_2)$

$B \rightarrow (v_1, \gamma_1), (v_2, \gamma_2)$

$$A = V \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} V^{-1}$$

$$B = V \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix} V^{-1}$$

$$BA\cancel{B}x = V \underbrace{\begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix}}_{\tilde{B}} \underbrace{\begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}}_{\tilde{A}} V^{-1}$$

Joint diagonalization

$A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n}$ are <u>jointly diagonalizable</u> if there exist an invertible V such that $\Lambda_1 = V^{-1}AV$, $\Lambda_2 = V^{-1}BV$ are both diagonal.

$$\Rightarrow AB = V\Lambda_1 V^{-1} V \Lambda_2 V^{-1} = V \underbrace{\Lambda_1 \Lambda_2}\, V^{-1}$$

$$= V \underbrace{\Lambda_2 \Lambda_1}_{V^{-1}V} V^{-1} = V\Lambda_2 V^{-1} V \Lambda_1 V^{-1} = BA$$

Eigen-decomposition

Assume $A \in \mathbb{R}^{n \times n}$ has $\underline{\underline{n}}$ $\underline{\text{linearly independent}}$ eigenvectors $\underbrace{v_1, v_2, \dots, v_n}_{\text{eigen basis}}$. $\quad Av_i = \lambda_i v_i$

$$V = [v_1 \ v_2 \cdots v_n]$$

$$AV = A[v_1 \ v_2 \cdots v_n] = [Av_1 \ Av_2 \cdots Av_n] = [\lambda_1 v_1 \ \lambda_2 v_2 \cdots \lambda_n v_n]$$

$$= \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}}_{\Lambda}$$

$\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$
diagonal matrix of eigenvalues

$$AV = V\Lambda$$

Eigen-decomposition

$$AV = V\Lambda$$

diagonal matrix of eigenvalues

$$\Lambda = V^{-1}AV$$

~~dia~~ diagonalization

A in the eigenbasis

A is diagonalizable iff A has an eigenbasis

($A \in \mathbb{R}^{n\times n}$ has n idependent eigen vectors)

A is diagonalizable

$$\Rightarrow AV = V\Lambda$$

$$\Lambda = V^{-1}AV$$

$$A = V\Lambda V^{-1}$$: eigen-decomposition

Eigen-decomposition

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A is diagonalizable eigen vectors

$$\Rightarrow AV = V\Lambda$$

$$\Lambda = V^{-1}AV$$

$$A = V\Lambda V^{-1}$$: eigen-decomposition

$$A^2 = AA = V\Lambda V^{-1}V\Lambda V^{-1} = V\Lambda^2 V^{-1} = V\begin{bmatrix}\lambda_1^2 & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2\end{bmatrix}V^{-1}$$

$$A^m = V\Lambda^m V^{-1}$$

$$A^m x = AAA\cdots Ax = V\Lambda^m \underbrace{V^{-1}x}_{\tilde{x}} = V\begin{bmatrix}\lambda_1^m \tilde{x}_1 \\ \lambda_2^m \tilde{x}_2 \\ \vdots \\ \lambda_n^m \tilde{x}_n\end{bmatrix}$$

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

$$Ax = \alpha_1\lambda_1 v_1 + \alpha_2\lambda_2 v_2 + \cdots + \alpha_n\lambda_n v_n \Rightarrow A^m x = \alpha_1\lambda_1^m v_1 + \alpha_2\lambda_2^m v_2 + \cdots + \alpha_n\lambda_n^m v_n$$

Eigen-decomposition

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$A^n x$

$\lim_{n\to\infty} A^n x = ?$

Assume $|\lambda_i| < 1$ for all eigenvalues λ_i.
and A is diagonalizable.

$\lim_{n\to\infty} A^n x = \lim_{n\to\infty} V \Lambda^n V^{-1} = V \lim_{n\to\infty} \Lambda^n V^{-1} = 0$

$\lim_{n\to\infty} A^n = 0_{n\times n}$

Eigen-decomposition

if $A \in \mathbb{R}^{n\times n}$ $\left(A \in \mathbb{C}^{n\times n}\right)$ has $\underline{\underline{n}}$ distinct eigenvalues (A has simple spectrum)

$\Longrightarrow$ A is diagonalizable

(the ~~reverse~~ converse is not necessarily true, $A = I$)

(Exercise)

[Geometric Vs. Algebraic Multiplicity]

Symmetric Matrices

All eigenvalues are real

Eigenvectors are orthogonal $v_i^T v_k = 0 \quad i \neq k$

$\Rightarrow$ scale v_i-s such that $\|v_i\| = 1$

$\Rightarrow v_i$'s are orthonormal

$$\Rightarrow V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \text{ is an orthogonal matrix} \qquad V^T V = I$$

$$V^{-1} = V^T$$

$$A = V \Lambda V^{-1} = V \Lambda V^T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$= \lambda_1 \underline{v_1 v_1^T} + \lambda_2 \underline{v_2 v_2^T} + \cdots + \lambda_n \underline{v_n v_n^T}$$

Positive Definiteness

$$\begin{cases} A^T = A \\ \forall x \neq 0 \quad \underbrace{x^T}_{1\times n}\underbrace{A}_{n\times n}\underbrace{x}_{n\times 1} > 0 \end{cases}$$

x^TAx: 1×1

$$Av = \lambda v \Rightarrow v^TAv = \lambda \underbrace{v^Tv}_{>0} > 0$$

$$\Rightarrow \lambda > 0$$

All eigenvalues > 0

A is positive definite $A \succ 0$

$$A \succ 0 \iff \lambda_i > 0 \quad \forall i$$

$$A = V\Lambda V^T = \Sigma\, \underline{\lambda_i}\, v_i \cdot v_i^T$$

$$x^TAx > 0$$

Positive semi-definite

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$x \in \mathbb{R}^n \neq 0 \qquad x^T A x = 0 \qquad$ A is singular

$A \succeq 0$

$x^T A x = 0$ at least one $\lambda_i = 0$

$$A = V \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} V^T = \underbrace{V \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & 0 & \\ & & & \lambda_4 \end{bmatrix} V}_{\text{rank} < n}$$

Positive semi-definite

$x^T A x \geqslant 0$ for all x. $\Rightarrow$ A is positive semi-definite