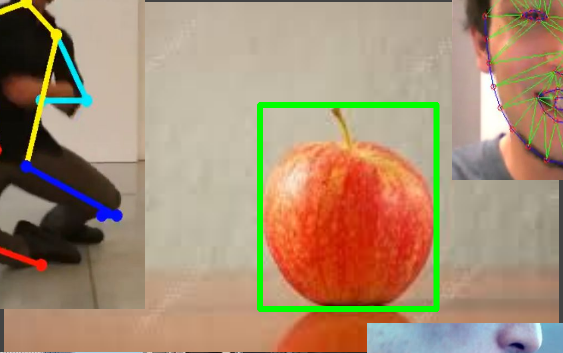
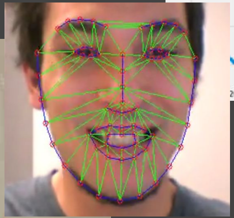
Mathematics for AI

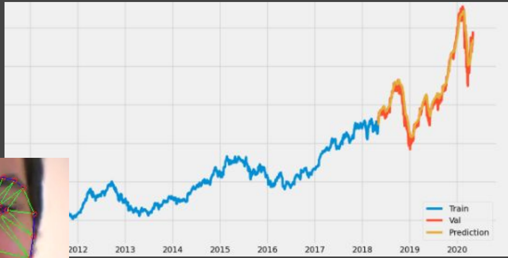
Lecture 2

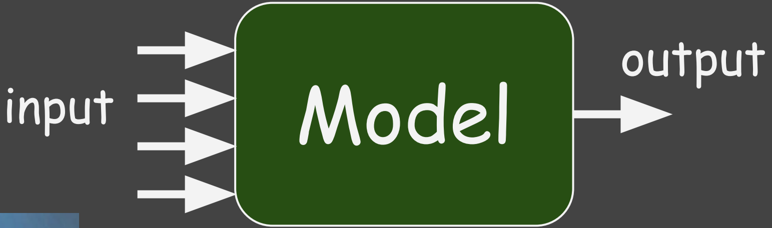
Vectors, Vector Space, Span, Basis, Coordinates

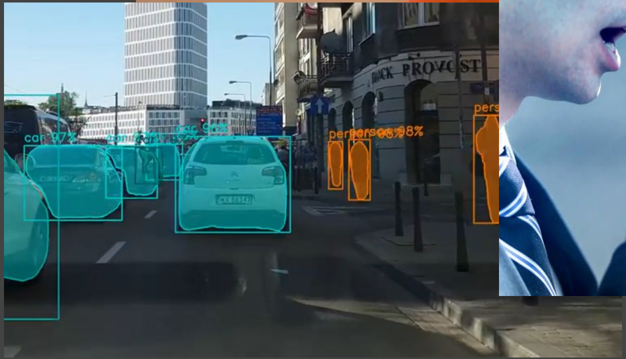
Machine Learning







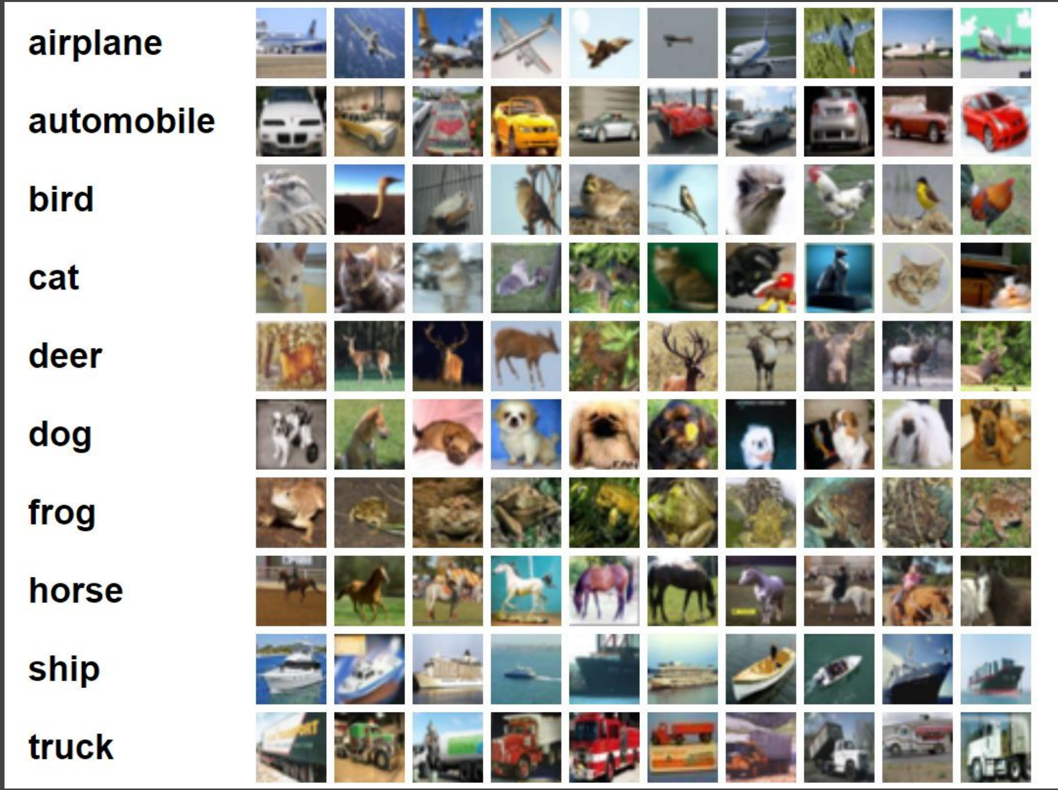




Learning from data

https://www.analyticsvidhya.com/blog/2018/03/comprehensive-collection-deep-learning-datasets/

Supervised Learning



http://seansoleyman.com/effect-of-dataset-size-on-image-classification-accuracy/

Supervised Learning

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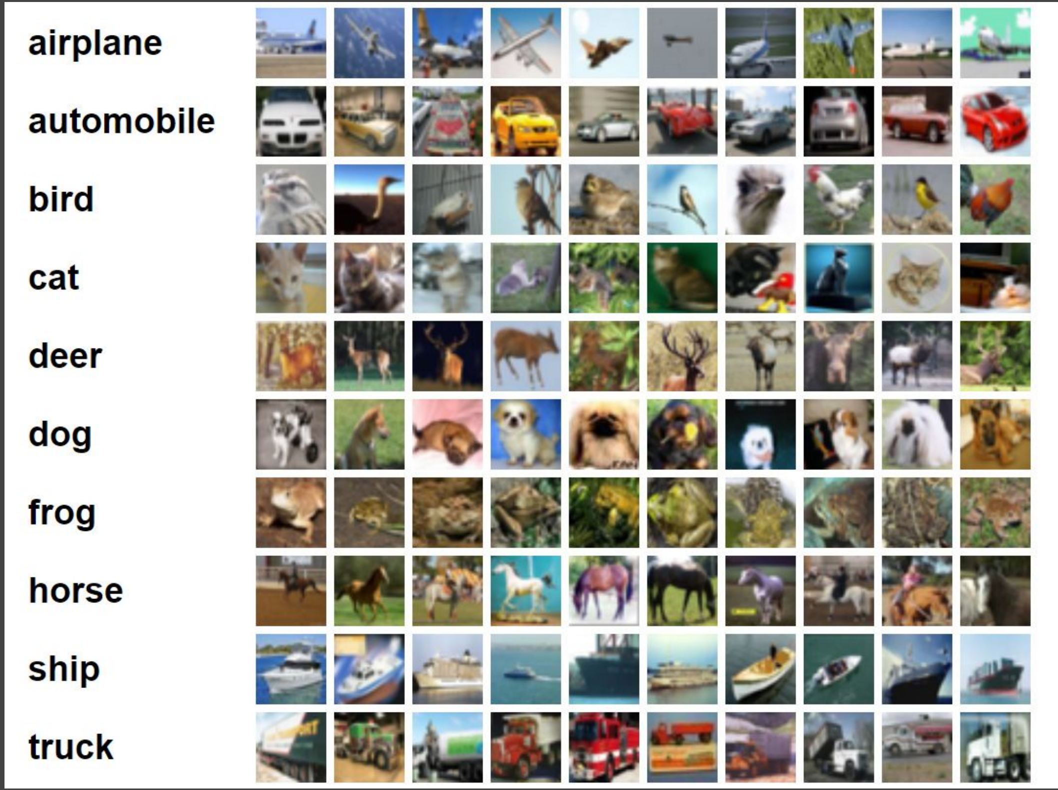
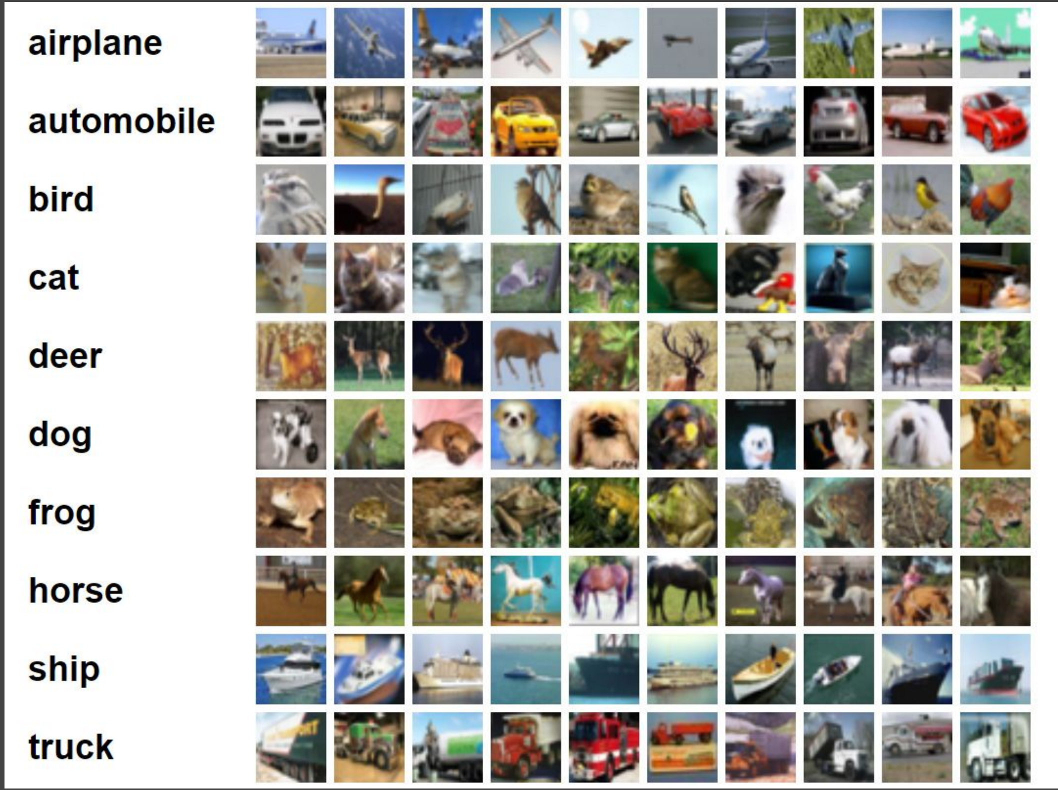


http://seansoleyman.com/effect-of-dataset-size-on-image-classification-accuracy/

Training data:

X_1, y_1
X_2, y_2
X_3, y_3
$\vdots$
X_n, y_n

Supervised Learning



http://seansoleyman.com/effect-of-dataset-size-on-image-classification-accuracy/

Training data:

	Apple
	Apple
	Orange
⋮	
	Orange

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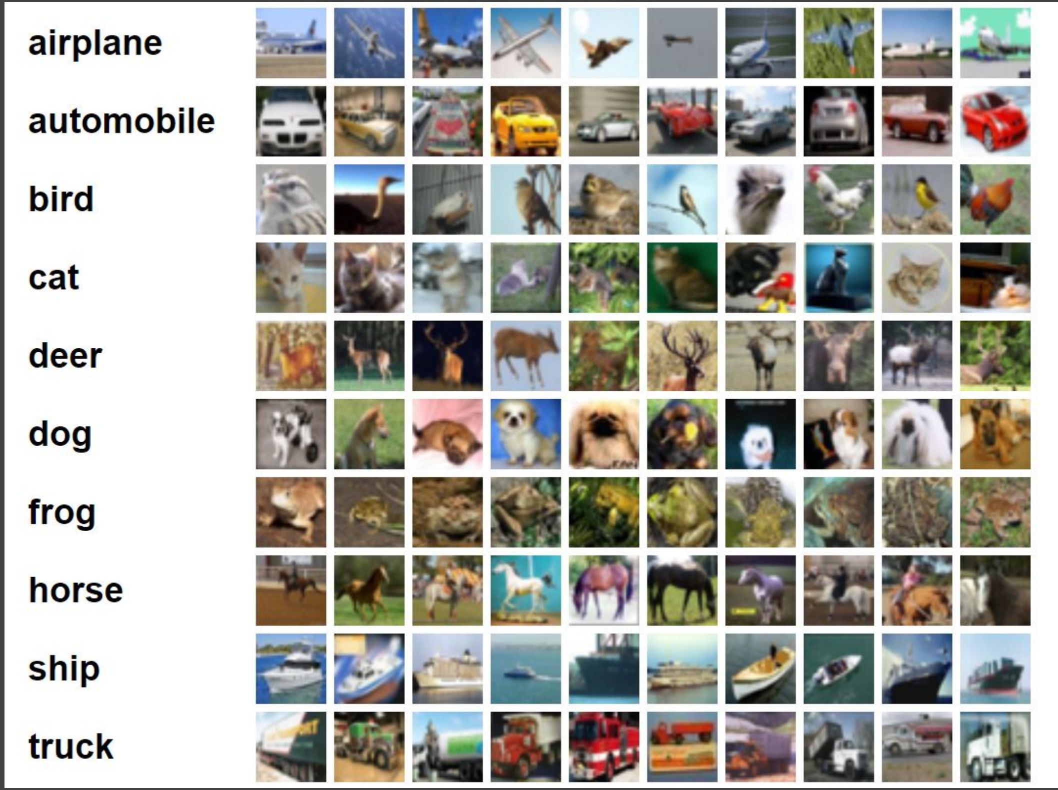


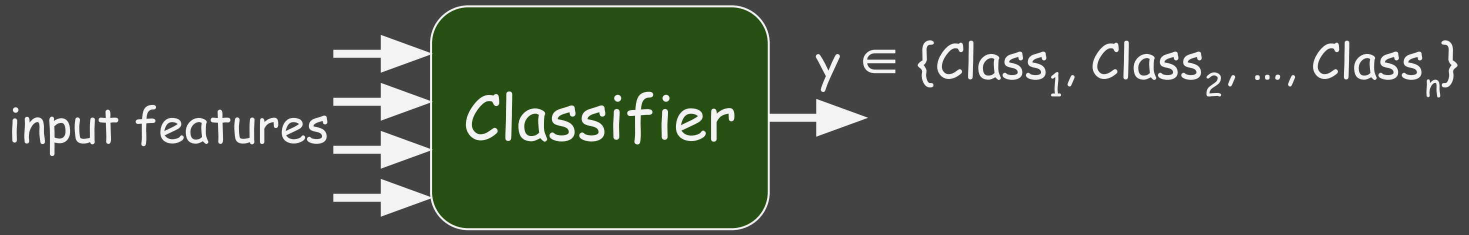
http://seansoleyman.com/effect-of-dataset-size-on-image-classification-accuracy/

Training data:

	0
	0
	1
⋮	
	1

Supervised Learning

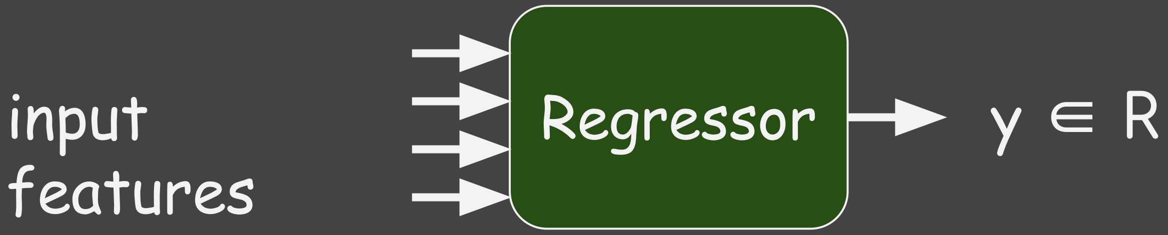
Classification



Classification

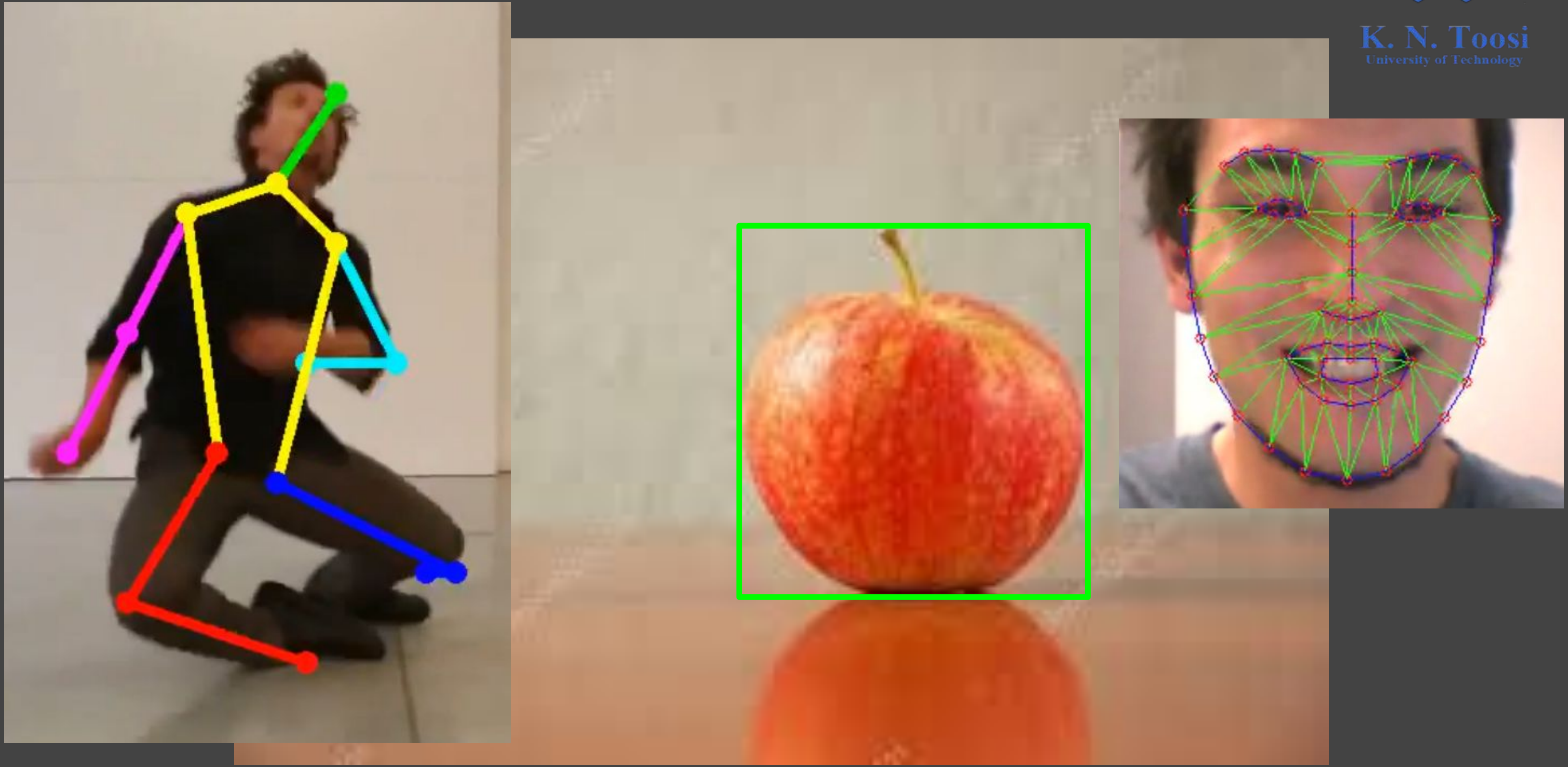
Classification

Regression



Regression

Regression



Learnable Models

Learnable Models: Example

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Learnable Models: Example

Learnable Models: Input-output map

$$y = f(x)$$

$$f: R^m \rightarrow R^n$$

Learnable Models: Input-output map

$$y = f(x,\theta)$$

$$f: R^m \rightarrow R^n$$

Learnable Models: Input-output map

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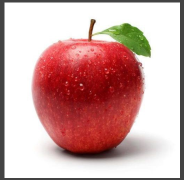
$y = f(x, \theta)$ $\quad \theta \in R^k$

$f: R^m \times R^k \rightarrow R^n$

Learnable Models: Example



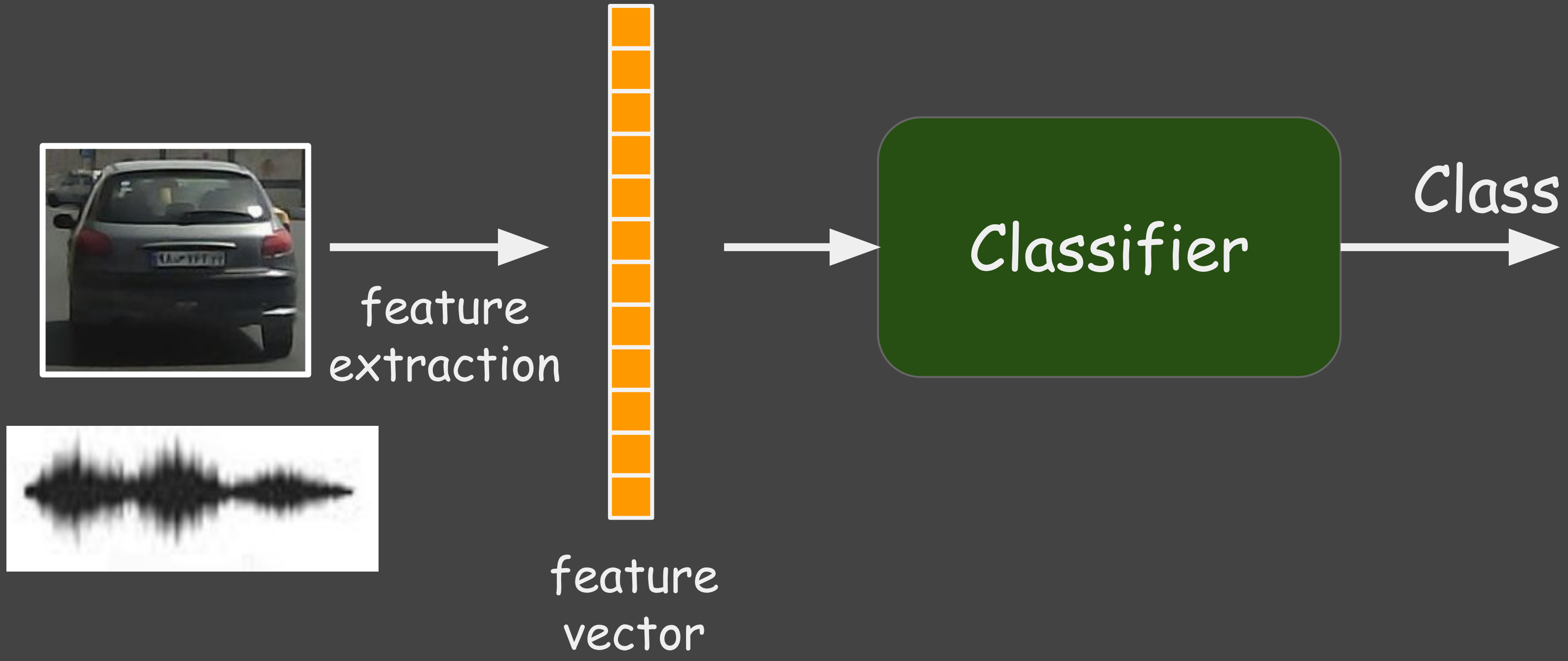
I

x = features(I) → f → y = 0

$$y = f(x, \theta)$$

$$f: R^m \times R^k \rightarrow R^n$$

Features



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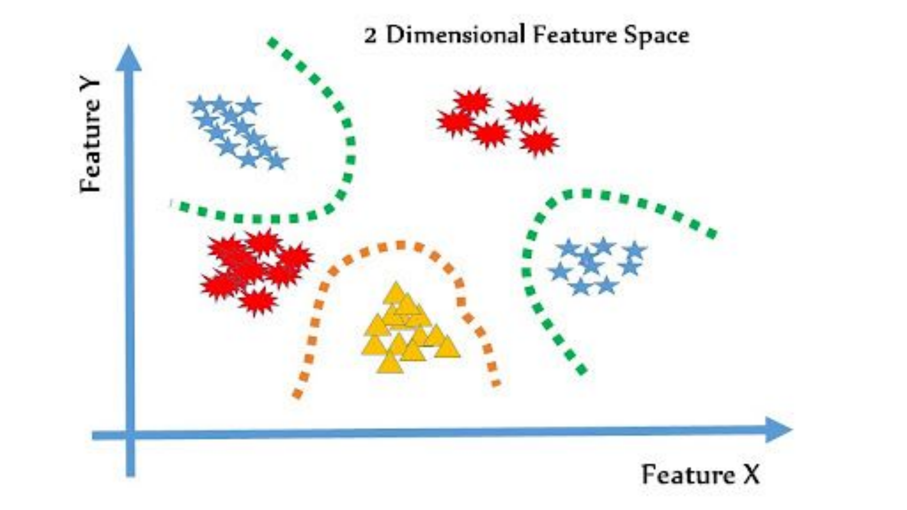
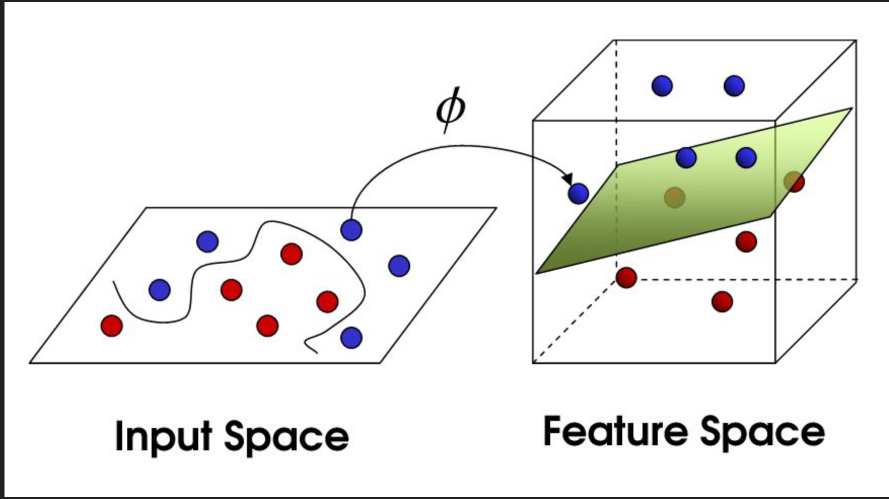
Feature space

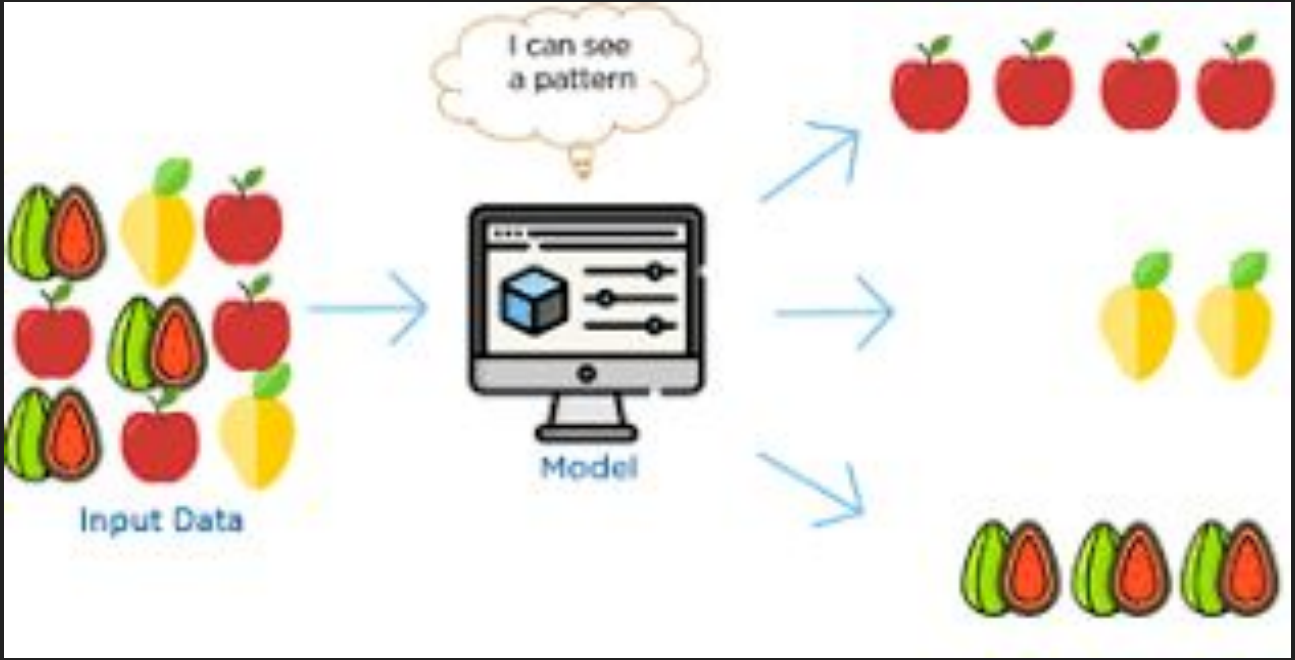


https://www.petersincak.com/news/why-i-do-not-believe-in-error-backpropagation/



https://towardsdatascience.com/the-kernel-trick-c98cdbcaeb3f

Unsupervised Learning



https://towardsdatascience.com/machine-learning-types-and-algorithms-d8b79545a6ec

Neural Networks

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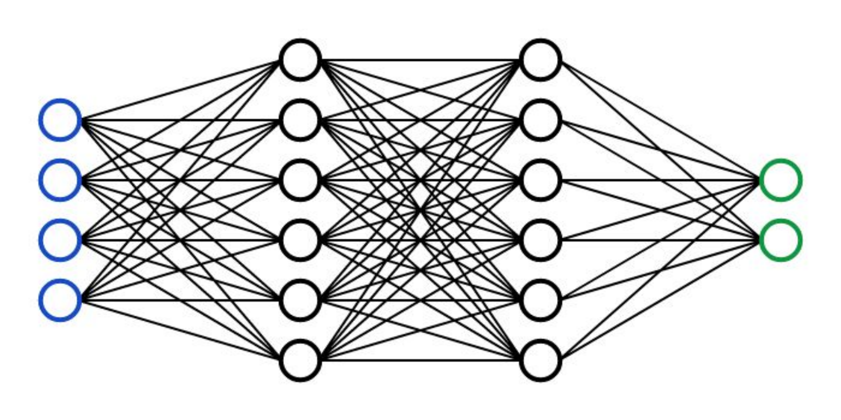
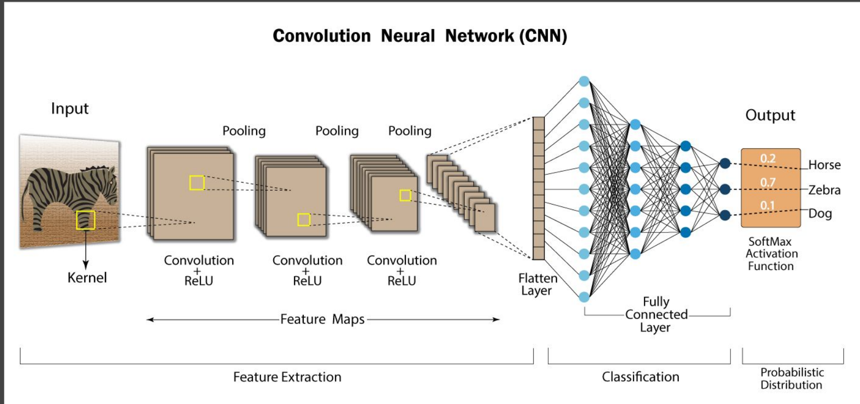
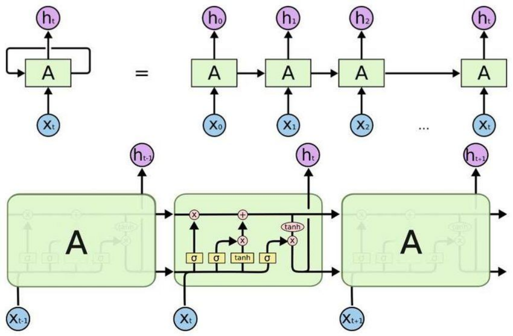
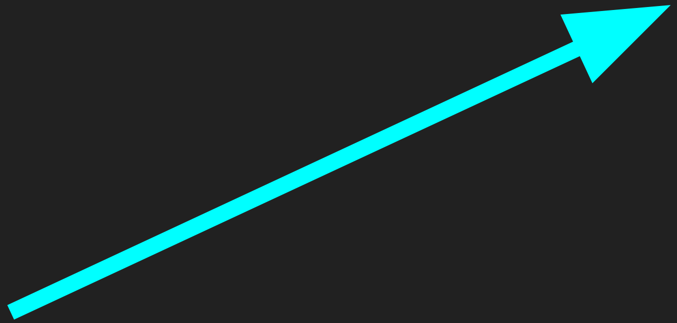






What is a Vector?



What is a Vector?

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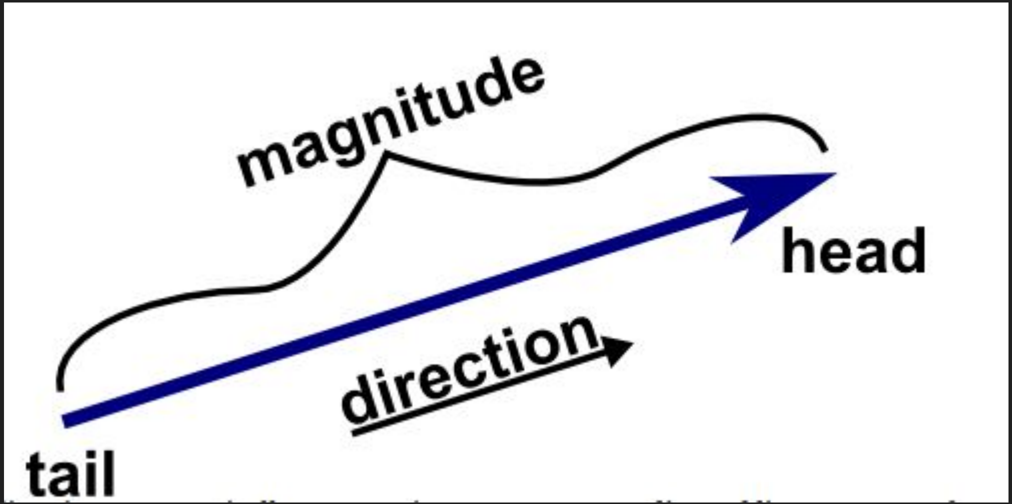
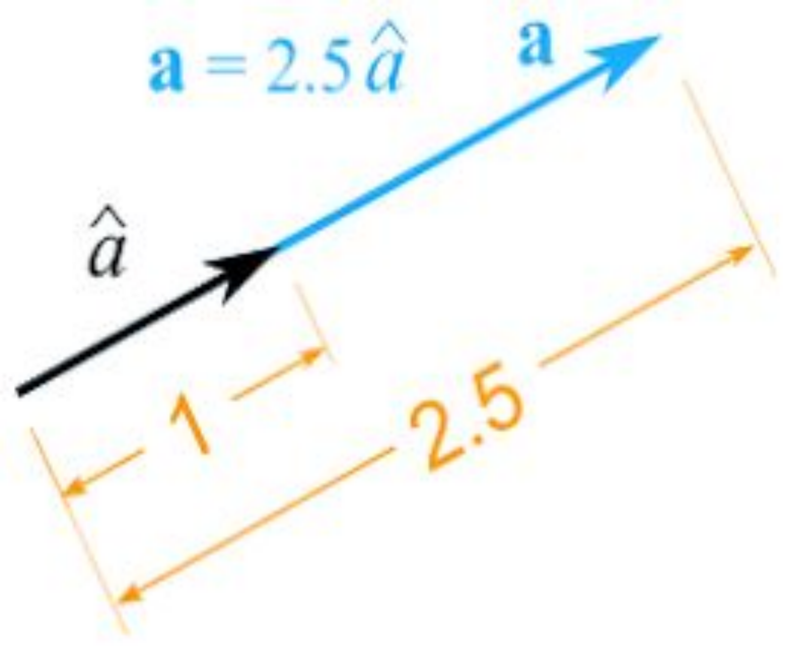
$$\vec{B} = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$



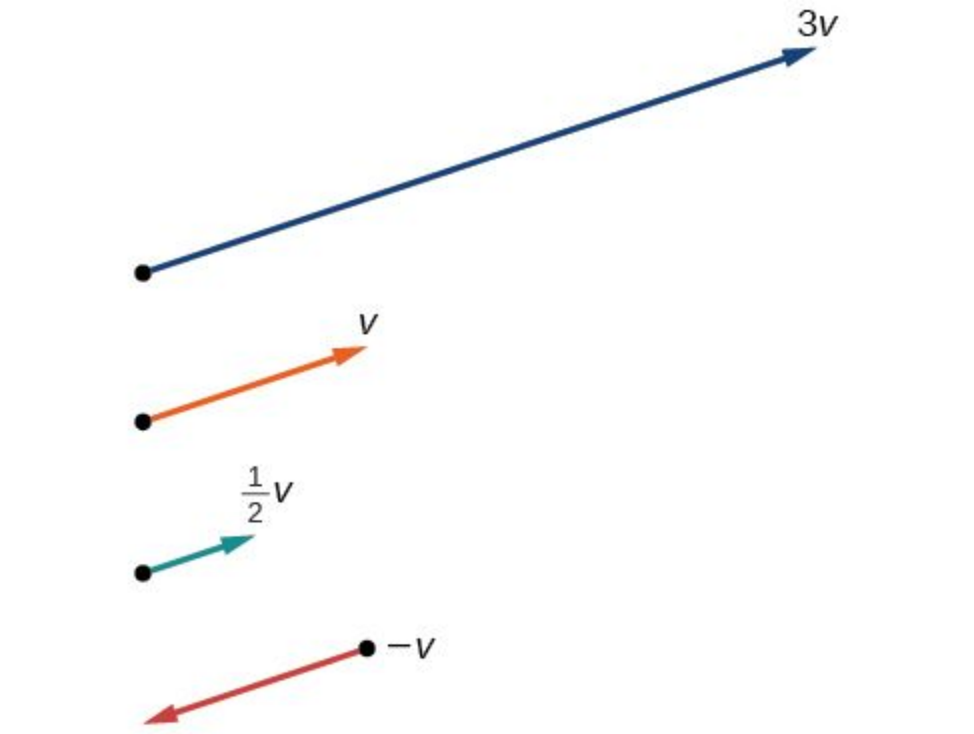
https://mathinsight.org/vector_introduction

Vector Scaling



https://semesters.in/unit-free-forced-fixed-vector/

Vector Scaling



https://philschatz.com/precalculus-book/contents/m49412.html

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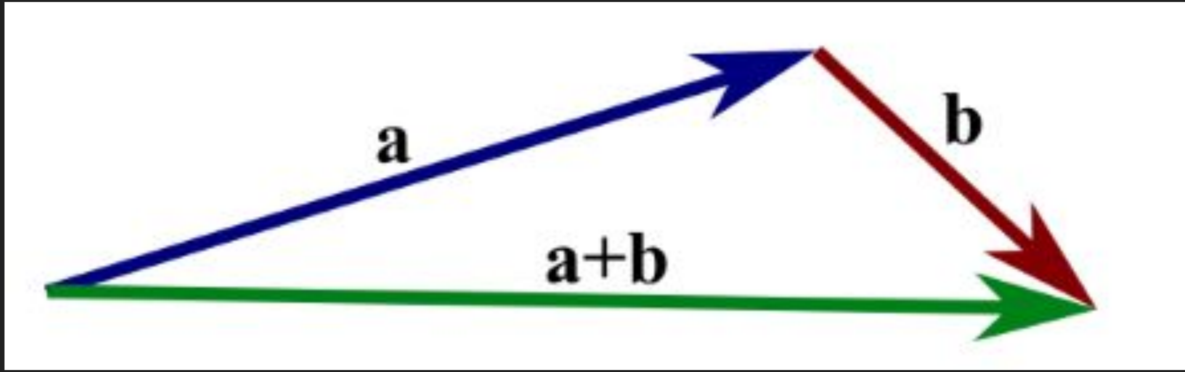
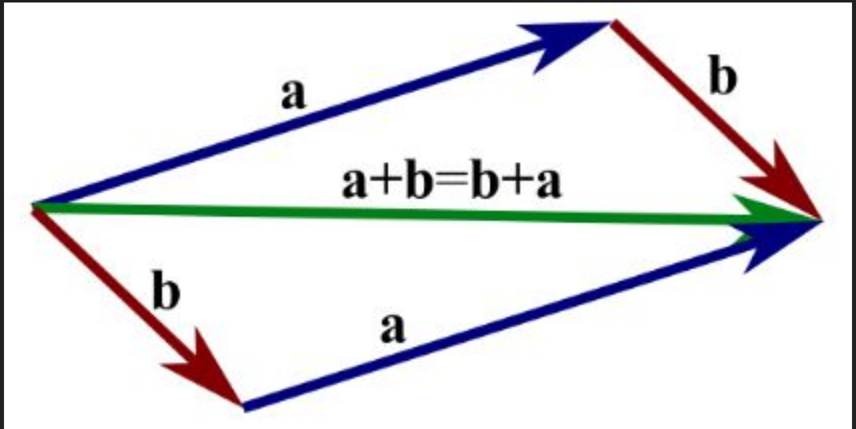
Vector Addition



https://mathinsight.org/vector_introduction

Vector Addition



https://mathinsight.org/vector_introduction

Space

A set with a **structure**

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Vector Spaces

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Vector Space

- a set V
- scalars ∈ R   (C, or any field)
- Vector addition +   (u + v for u,v ∈ V)
- scalar multiplication (a u   for a ∈ R, u ∈ V)
  - Commutativity: u + v = v + u
  - Associativity: u + (v + w) = (u + v) + w
  - Identity element: ∃ z ∈ V : v + z = z + v = v
  - Inverse: for each v ∈ V there is v' : v + v' = z   (z defined above)
  - (ab) v = a (b v)
  - 1 v = v
  - a (u + v) = a u + a v
  - (a+b) v = a v + b v

Why bother?

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Why bother? adding apples and pears?

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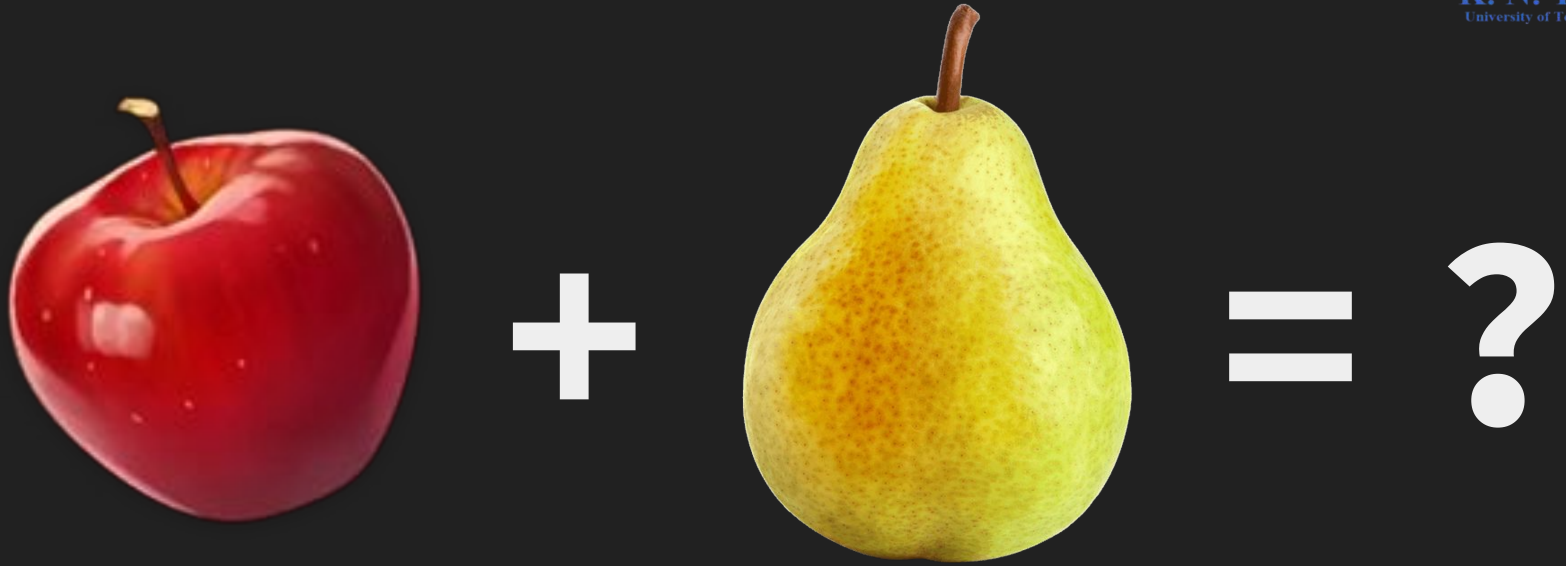
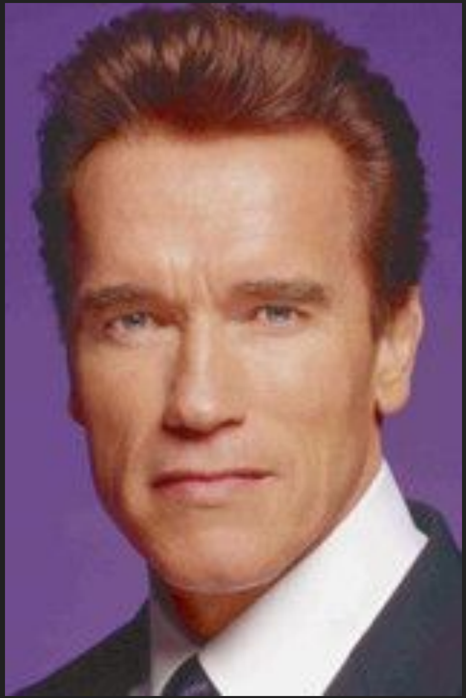


Why bother? Shape+Appearance Averaging

Jerk



Cyborg

Why bother? Shape+Appearance Averaging

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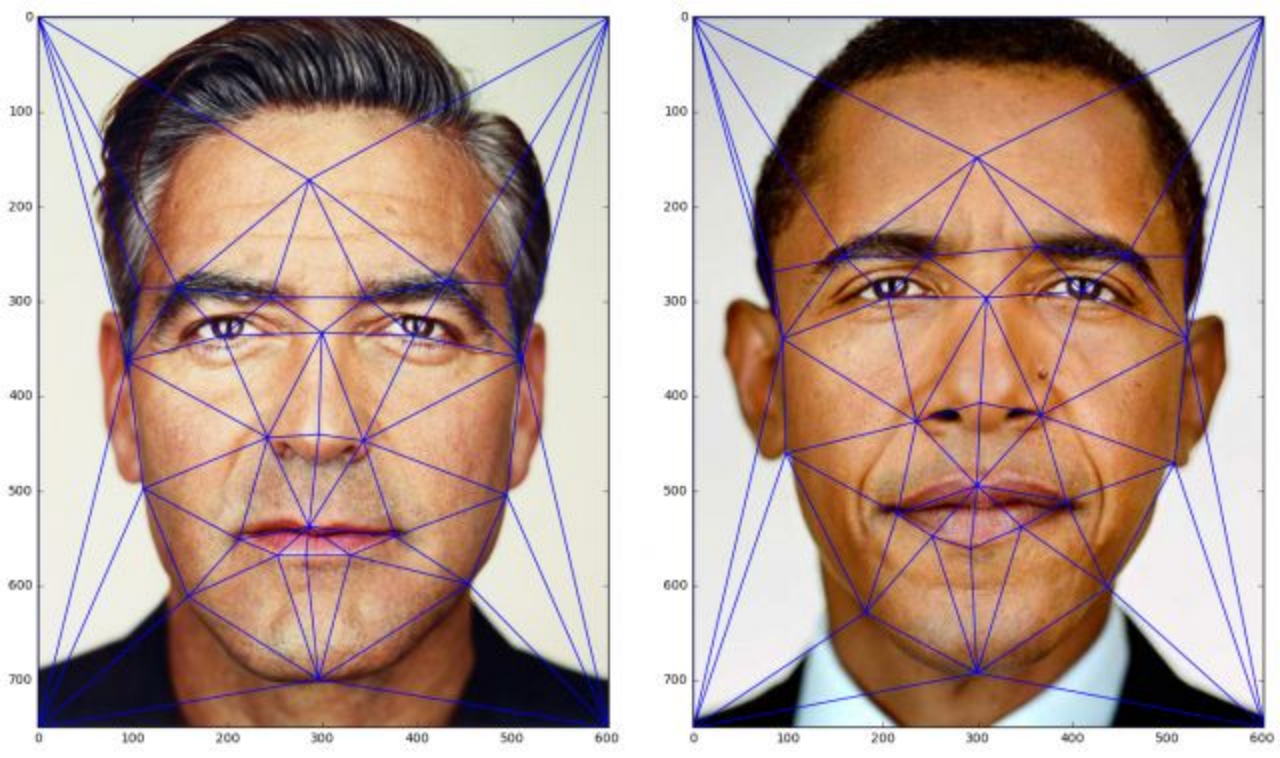
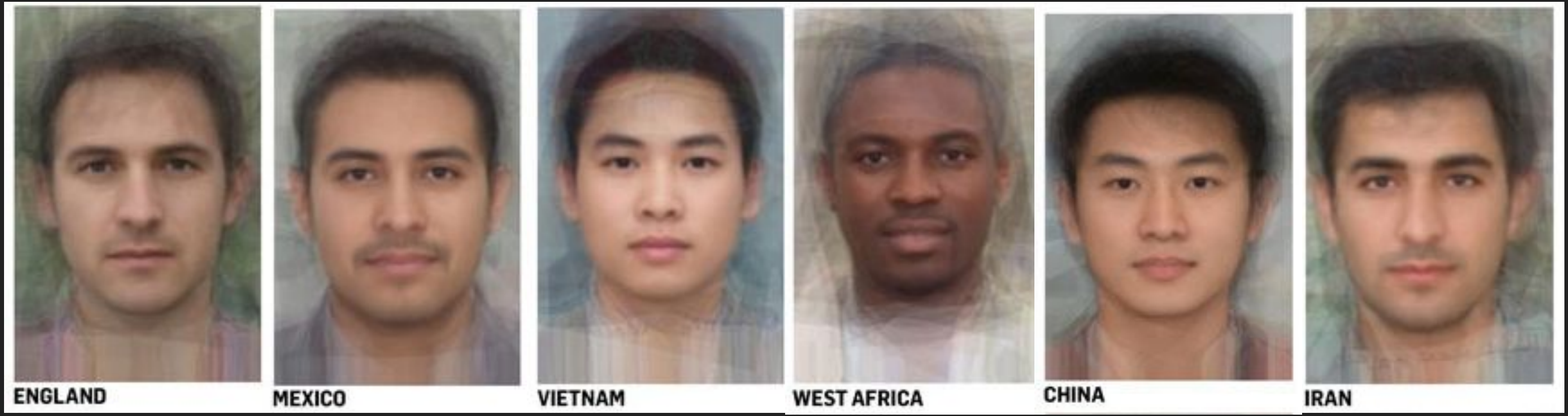
Jerk Cyjerk Cyborg

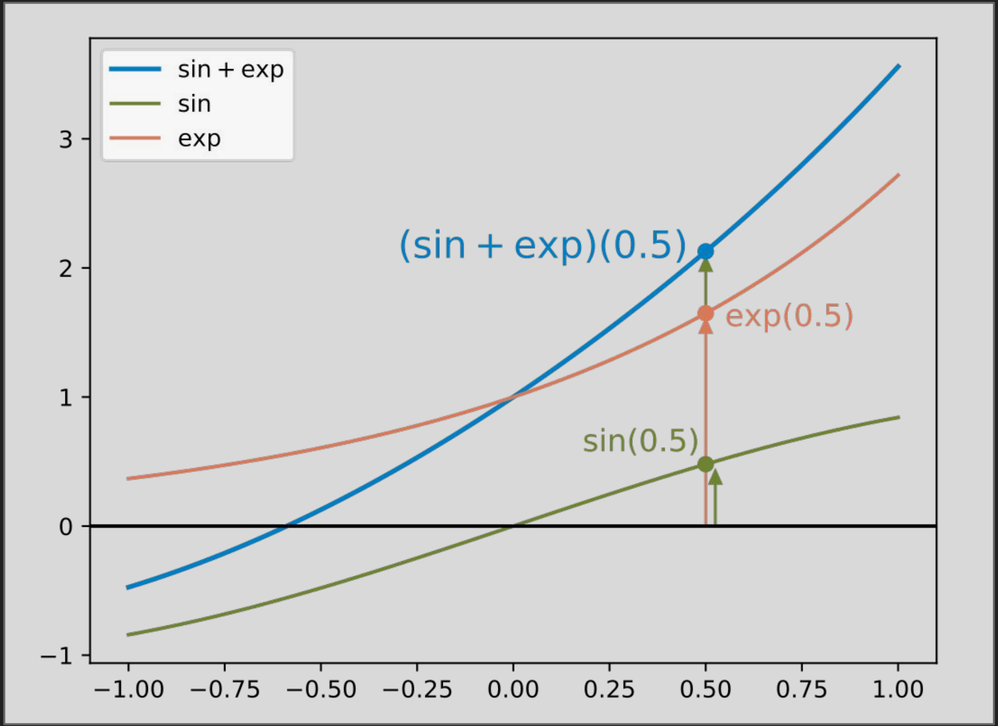
Why bother? Define vector addition and scaling



Why bother? Average Faces by country

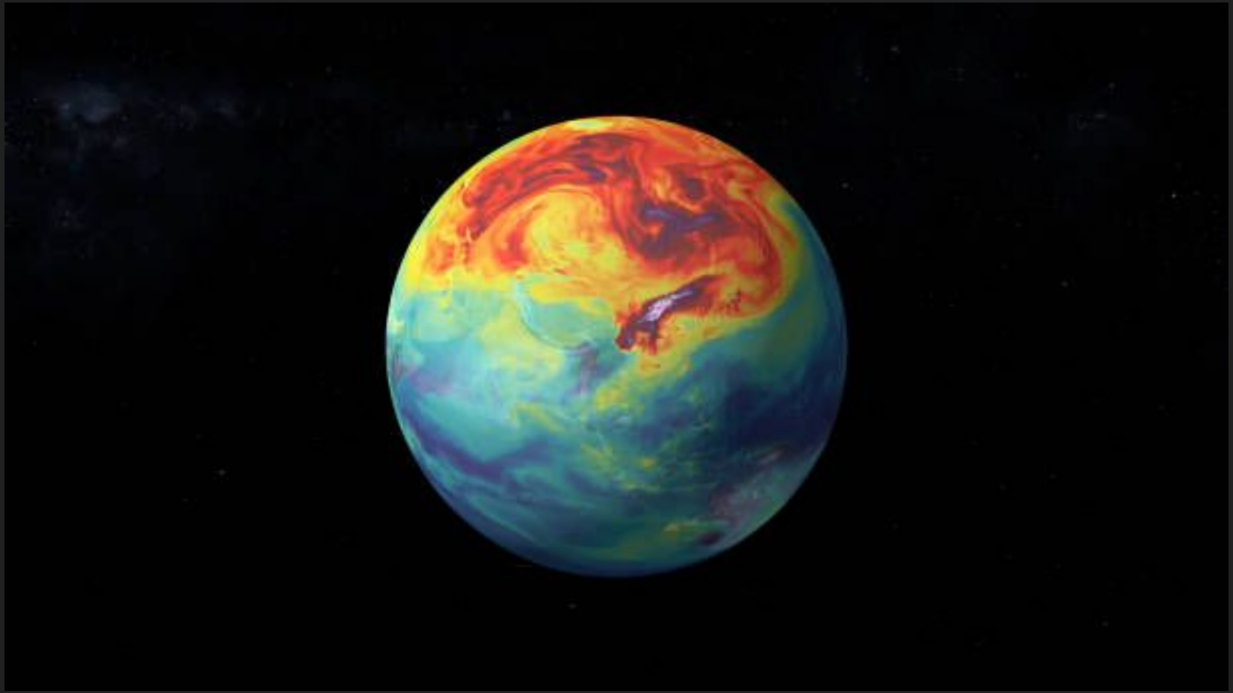


Why bother? functions as vectors



https://en.wikipedia.org/wiki/Vector_space

Why bother? functions as vectors



Linear combination

Let $a, b \in R$. The vector $a\,x + b\,y$ is a linear combination of the vectors x and y.

Let $a_i \in R$. The vector $a_1 x_1 + a_2 x_2 + \dots + a_n x_n$ is a linear combination of the vectors $x_1, x_2, \dots, x_n$.

Span

$\text{span}(x,y) = \{ a x + b y \mid a,b \in R\}$

The space of all linear combinations of x and y.

$\text{span}(x_1, x_2, \ldots, x_n) = \{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \mid a_i \in R \}$

Span

We say that $x_1, x_2, \ldots, x_n$ span S if $S = \mathrm{span}(x_1, x_2, \ldots, x_n)$.

Linear dependence

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x,y,z are dependent if

- $x \in \text{span}(y,z)$, OR
- $y \in \text{span}(z,x)$, OR
- $z \in \text{span}(x,y)$

that is

- $x = a\,y + b\,z$, for some a, b, OR
- $y = a\,z + b\,x$, for some a,b, OR
- $z = a\,x + b\,y$, for some a,b.

Linear dependence

$x_1, x_2, \dots, x_n \in V$ are **linearly dependent** if one of them can be written as a linear combination of the others (one of them is in the span of the others).

Linear dependence

$x_1, x_2, \ldots, x_n \in V$ are **linearly dependent** if one of them can be written as a linear combination of the others (one of them is in the span of the others).

Linear independence

x,y,z are independent if

- $x \notin \text{span}(y,z)$, AND
- $y \notin \text{span}(z,x)$, AND
- $z \notin \text{span}(x,y)$

Linear independence

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$x_1, x_2, \ldots, x_n \in V$ are **linearly independent** if none of them can be written as a linear combination of the others.

Linear independence

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$x_1, x_2, \ldots, x_n \in V$ are **linearly independent** if none of them can be written as a linear combination of the others.

Equivalently:

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0 \quad \Rightarrow \quad a_1 = a_2 = \ldots = a_n = 0$$

Basis

$v_1, v_2, \ldots, v_n \in V$ such that

- $v_1, v_2, \ldots, v_n$ are linearly independent
- $v_1, v_2, \ldots, v_n$ span V

Basis

$v_1, v_2, \ldots, v_n \in V$ such that

- $v_1, v_2, \ldots, v_n$ are linearly independent
- $v_1, v_2, \ldots, v_n$ span V

* n is the same for any choice of the basis vectors

* n is called the dimension of V

* There are also **infinite dimensional** vector spaces

* Basis (general definition)

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$\{v_i\}_{i\in I} \subseteq V$ such that

- v_i's are linearly independent
- for any $v \in V$ there is a **finite** set of vectors $v_1, v_2, \ldots, v_d \in \{v_i\}_{i\in I}$ such that $\mathbf{v} \in \mathbf{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d)$

* Any vector space has a basis

* cardinality of $\{v_i\}_{i\in I}$ is the same for any choice of the basis vectors

* cardinality of $\{v_i\}_{i\in I}$ is called the dimension of V

Bases and Coordinate Representation

Why is independence needed? => uniqueness

every $x \in V$ can be written **uniquely** as a linear combination of the basis vectors $v_1, v_2, \ldots, v_n$.

Bases and Coordinate Representation

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⇒ Every $x \in V$ can be written as a unique linear combination of $u_1, \ldots, u_n$.

$$x = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$$

⇒ x can be represented as $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$.

$$x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$ as an array of real numbers.

a_i-s are called coordinates of x

مختصات

مختصات به بردارهای پایه وابسته است

Example: The Euclidean space