Mathematics for AI

Lecture 20

Modeling Uncertainty, Random Variables, Probability mass and density functions,

Mathematical Tools for modeling

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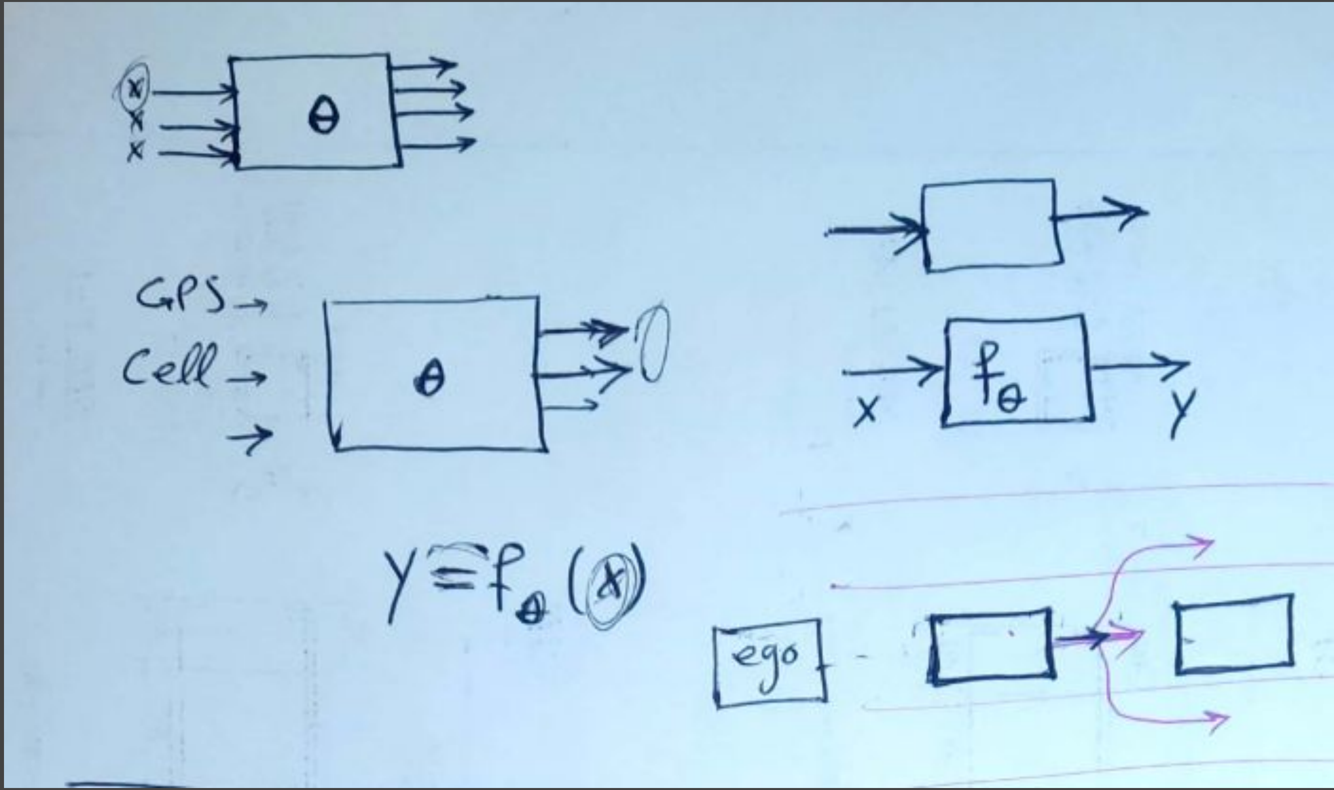
- Variables
- Functions, Time series, Signals & Systems
- (ordinary, differential, integral) equations

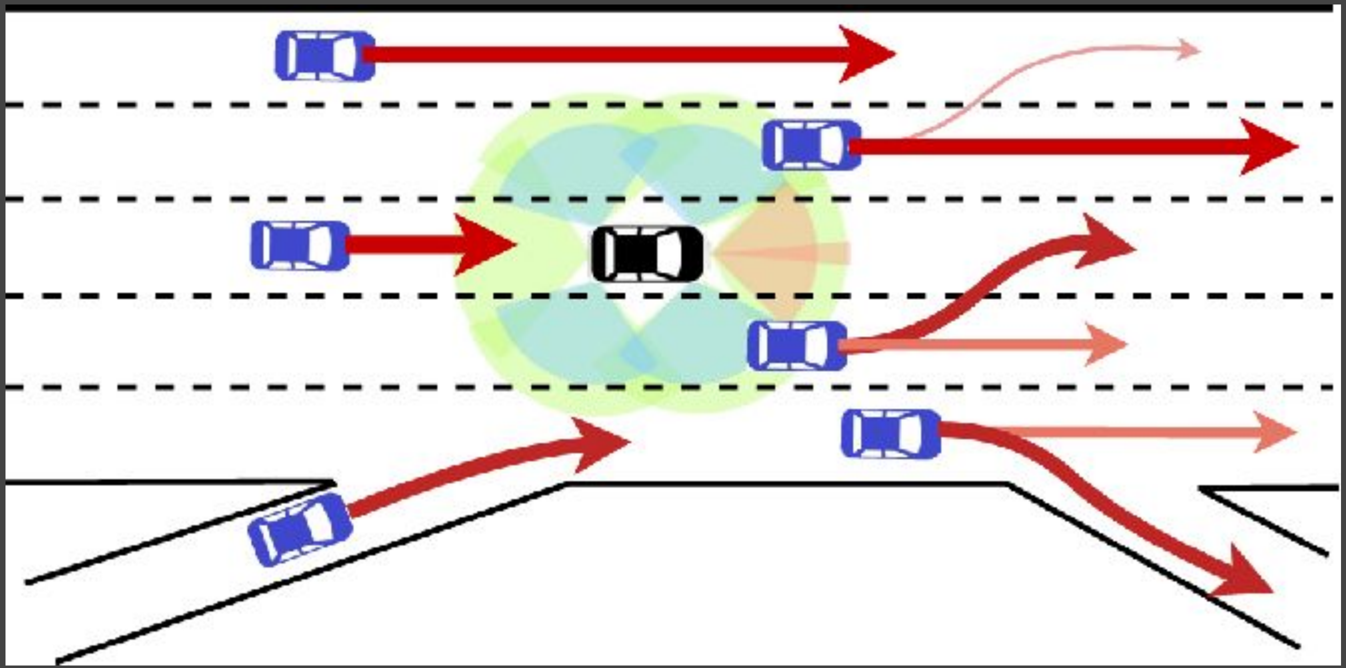
Example:

$$\mathbf{x} = \frac{\mathbf{f}}{2m} t^2 + \mathbf{v}_0 t + \mathbf{x}_0$$

How to model uncertainty



How to model uncertainty



https://www.semanticscholar.org/paper/Multi-Modal-Trajectory-Prediction-of-Surrounding-Deo-Trivedi/305c4d91b0f70853a1cb0ed2a60a466b84e5c13d

Uncertainty

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- Problems including uncertainty, hidden variables,
- Unknown factors,
- All factors cannot be exactly measured,
- Complex factors or relations that are too hard to model

Example:

$$\mathbf{x} = \frac{\mathbf{f}}{2m}t^2 + \mathbf{v}_0 t + \mathbf{x}_0$$

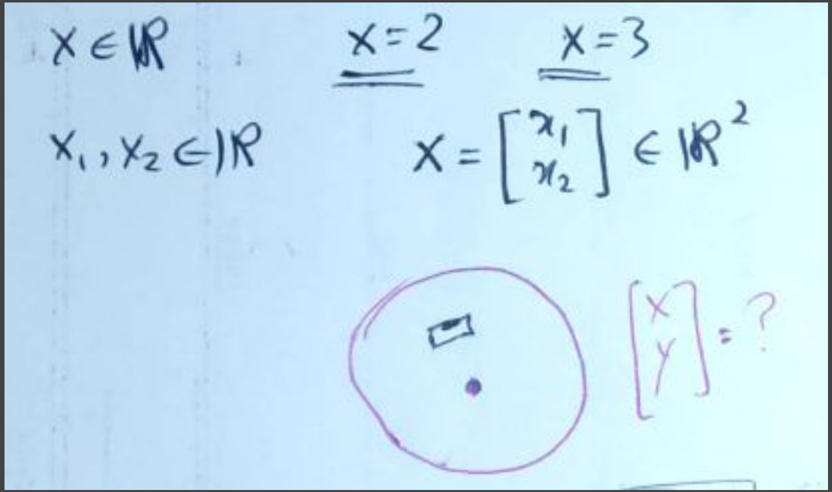
where

$$\mathbf{f} = \mathbf{f}_{\text{known}} + \mathbf{f}_{\text{wind}}$$

Uncertainty

- Problems including uncertainty, hidden variables,
- Unknown factors,
- All factors cannot be exactly measured,
- Complex factors or relations that are too hard to model

Example: GPS



Uncertainty

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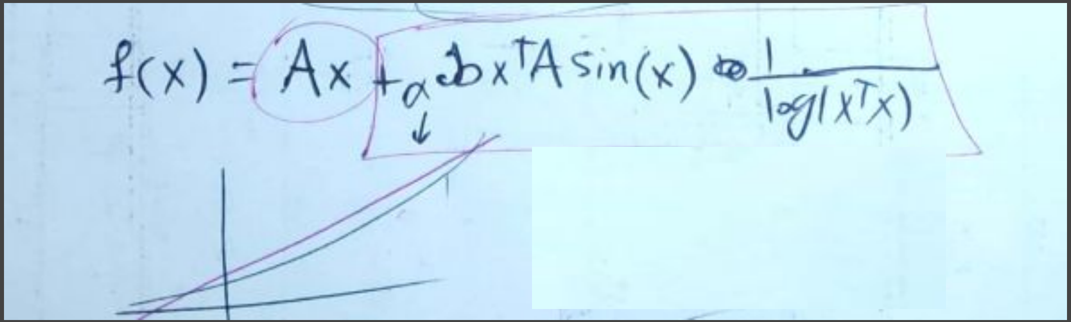
- Problems including uncertainty, hidden variables,
- Unknown factors,
- All factors cannot be exactly measured,
- **Complex factors or relations that are too hard to model**

Example:



Probabilistic Models

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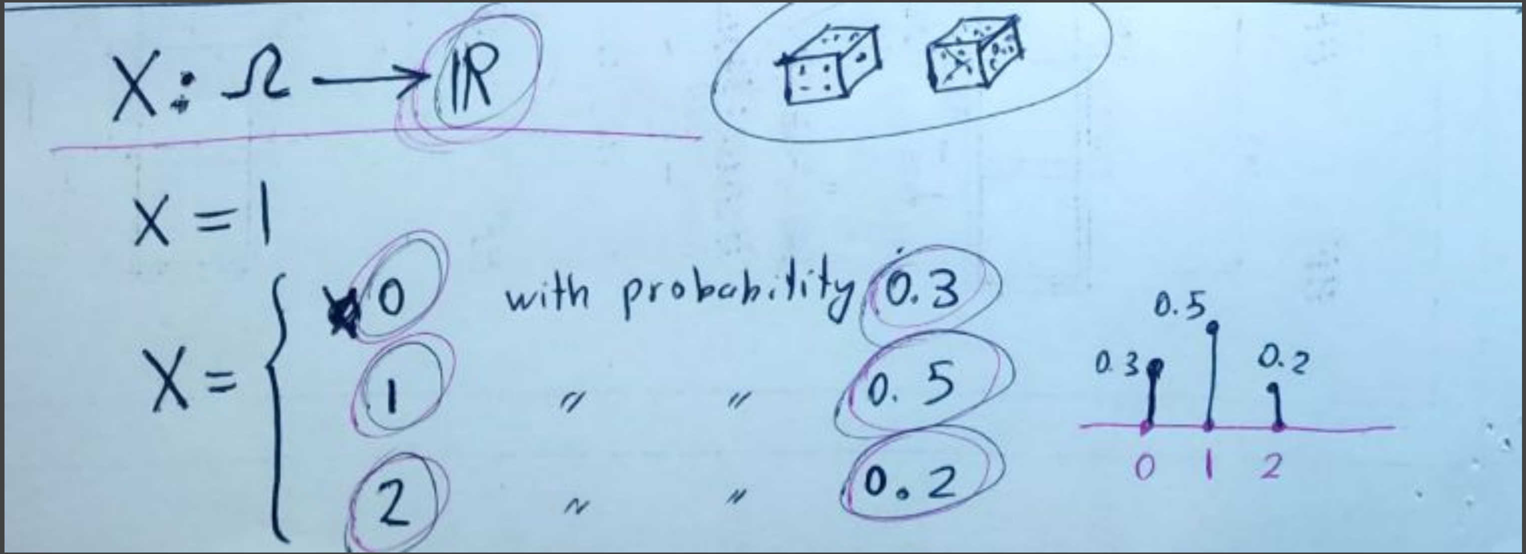
- Model probabilistic uncertainty/randomness
- Random Variables
- Distributions

Random Variables



Random Variables

- Profit = Revenue – Cost
  - Revenue = f(n)
  - Cost = g(n)
  - n : number of packages produced per day
- Ordinary (deterministic) Variables:
  - n = 244
- Random Variables:
  - n = 243 with a probability of 1/6
  - n = 244 with a probability of 3/6
  - n = 245 with a probability of 2/6

Random Variables

- Ordinary (deterministic) Variables:
  - n = 244
- Random Variables:
  - n = 243 with a probability of 1/6
  - n = 244 with a probability of 3/6
  - n = 245 with a probability of 2/6
- Mathematical Operations
  - addition, subtraction, multiplication, etc.
  - functions
  - reasoning

Representing Random Variables

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- Random Variables:
  - n = 243 with a probability of 1/6
  - n = 244 with a probability of 3/6
  - n = 245 with a probability of 2/6

- Probability Distribution
- Probability Mass Function
  - p = {(243, 1/6), (244, 3/6), (245, 2/6)}
  - p(243) = 1/6

Representing Random Variables

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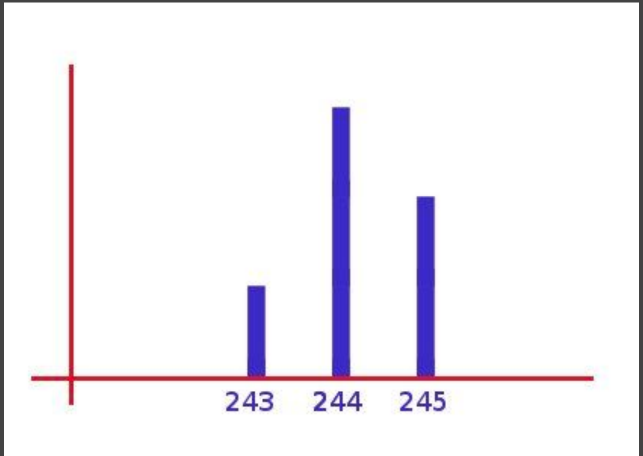
- Random Variables:
  - n = 243 with a probability of 1/6
  - n = 244 with a probability of 3/6
  - n = 245 with a probability of 2/6

- Probability Distribution
- Probability Mass Function
  - p = {(243, 1/6), (244, 3/6), (245, 2/6)}
  - p(243) = Pr(N=243) = 1/6



Representing Random Variables

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- Random Variables:
  - n = 243 with a probability of 1/6
  - n = 244 with a probability of 3/6
  - n = 245 with a probability of 2/6

- Ordinary variables as special case of random variables:

Representing Random Variables

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- Random Variables:
  - n = 243 with a probability of 1/6
  - n = 244 with a probability of 3/6
  - n = 245 with a probability of 2/6

- Ordinary variables as special case of random variables:
  - n = 243 with a probability of 0
  - n = 244 with a probability of 1
  - n = 245 with a probability of 0

Operations on random variables

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$$X = \begin{cases} 0 & p = 0.3 \\ 1 & p = 0.5 \\ 2 & p = 0.2 \end{cases} \qquad 2X = ?$$

$$Y = f(X)$$

$$Y = 2X \qquad Y = \begin{cases} 0 & p = 0.3 \\ 2 & p = 0.5 \\ 4 & p = 0.2 \end{cases}$$

$$Z = X^2 \qquad Z = \begin{cases} 0 & p = 0.3 \\ 1 & p = 0.5 \\ 4 & p = 0.2 \end{cases}$$

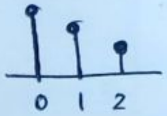
$$T = (X-1)^2 \qquad T = \begin{cases} 0 & p = 0.5 \\ 1 & p = 0.3 + 0.2 = 0.5 \end{cases}$$

Adding two independent random variables

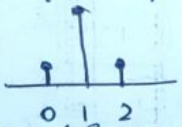
X: How many products Mehran will sell today.

Y: " " " Milad " " ".

$$X = \begin{cases} 0 & p=0.5 \\ 1 & p=0.3 \\ 2 & p=0.2 \end{cases}$$



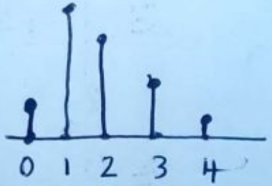
$$Y = \begin{cases} 0 & p=0.1 \\ 1 & p=0.8 \\ 2 & p=0.1 \end{cases}$$



Z: How many products Milad & Mehran will sell today.

$Z = X + Y$ (Assuming X and Y are independent)

$$Z = \begin{cases} 0 & p=0.5\times0.1=0.05 \\ 1 & p=0.5\times0.8+0.3\times0.1=0.43 \\ 2 & p=0.2\times0.1+0.3\times0.8+0.5\times0.1=0.31 \\ 3 & p=0.2\times0.8+0.1\times0.3=0.19 \\ 4 & p=0.2\times0.1=0.02 \end{cases}$$

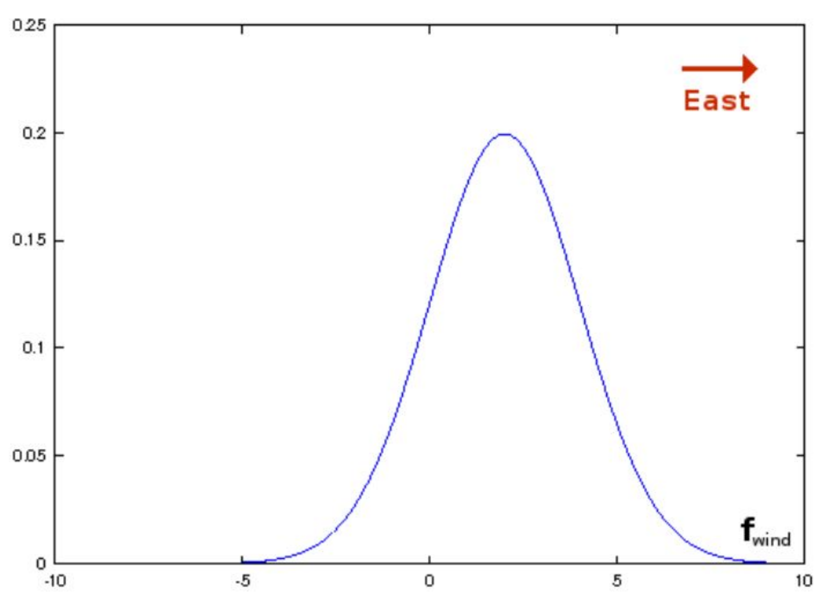


Continuous Random Variables

- $\mathbf{x} = \frac{\mathbf{f}}{2m} t^2 + \mathbf{v}_0 t + \mathbf{x}_0$
- $\mathbf{f} = \mathbf{f}_{\text{known}} + \mathbf{f}_{\text{wind}}$
- $\mathbf{f}_{\text{wind}} = ?$

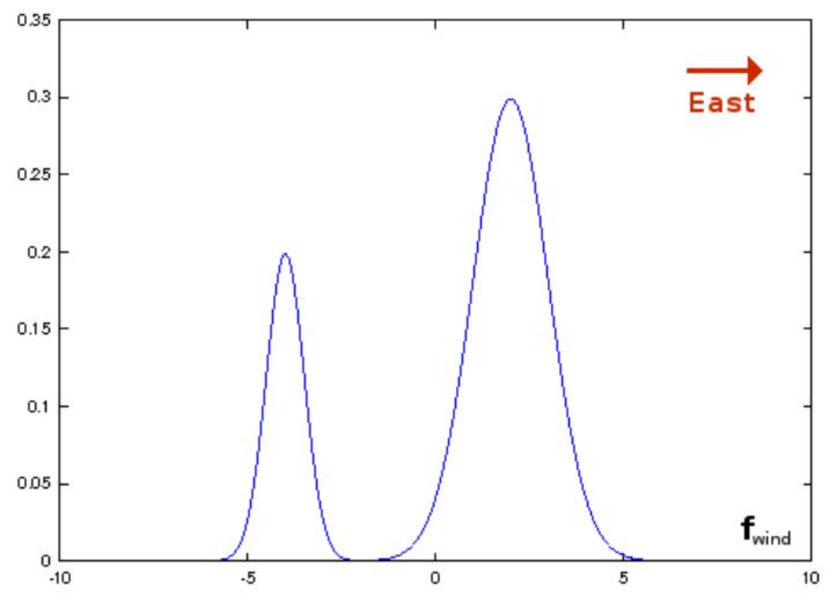
Continuous Random Variables

- $\mathbf{x} = \frac{\mathbf{f}}{2m} t^2 + \mathbf{v}_0 t + \mathbf{x}_0$
- $\mathbf{f} = \mathbf{f}_{\text{known}} + \mathbf{f}_{\text{wind}}$
- $\mathbf{f}_{\text{wind}} = ?$



Continuous Random Variables

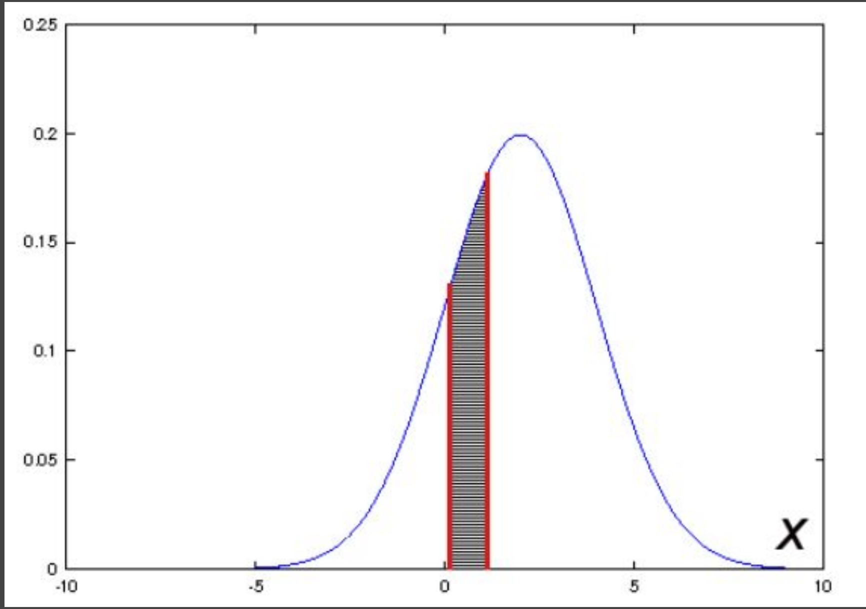
- $\mathbf{x} = \frac{\mathbf{f}}{2m} t^2 + \mathbf{v}_0 t + \mathbf{x}_0$
- $\mathbf{f} = \mathbf{f}_{\text{known}} + \mathbf{f}_{\text{wind}}$
- $\mathbf{f}_{\text{wind}} = ?$



Continuous Random Variables

- the probability of a variable being in a certain interval is of interest,
- Probability Density Function (PDF)

$$\Pr(\mathrm{a} \le \mathrm{x} \le \mathrm{b}) = \int_{x \in (a,b)} p(x)dx$$



Continuous Random Variables

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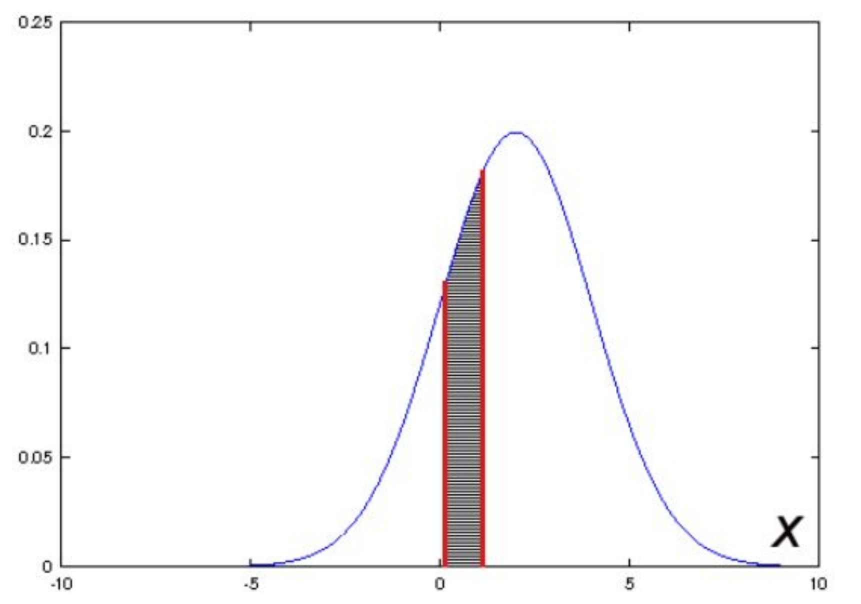
- In continuous case, mostly the probability of a variable being in a certain interval is of interest,
- Probability Density Function (PDF)

$$\Pr(\mathrm{a} \leq \mathrm{x} \leq \mathrm{b}) = \int_{x \in (a,b)} p(x) dx$$

$$\Pr(x \in S) = \int_S p(x) dx$$



Probability Mass Function vs Probability Density Function

PMF $\quad p(n) = \Pr(N = n) \quad n \in \mathbb{Z}$

PDF $\quad p(x) \quad \int_a^b p(x)\,dx = \Pr(a \leq X \leq b) \quad x \in \mathbb{R}$

Cumulative Distribution

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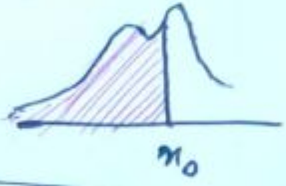
$$P(n_0) = Pr(N \leqslant n_0) = \sum_{n=-\infty}^{n_0} p(n)$$

$$P(x_0) = Pr(X \leqslant x_0) = \int_{-\infty}^{x_0} p(x)\,dx$$



Adding two random variables (general case)

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