Mathematics for AI

Lecture 22

Probabilistic Independence,

Question

- Pr(rain in 1hr) = .15
- Pr(rain in 1hr | cloudy now) = .5

- Pr(rain in 1hr) = .15
- Pr(rain in 1hr | I failed the math exam) = ?

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Probabilistic Independence

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- What does independence mean?
  - does "having a rainfall" depend on "people using umbrellas"?

Probabilistic Independence

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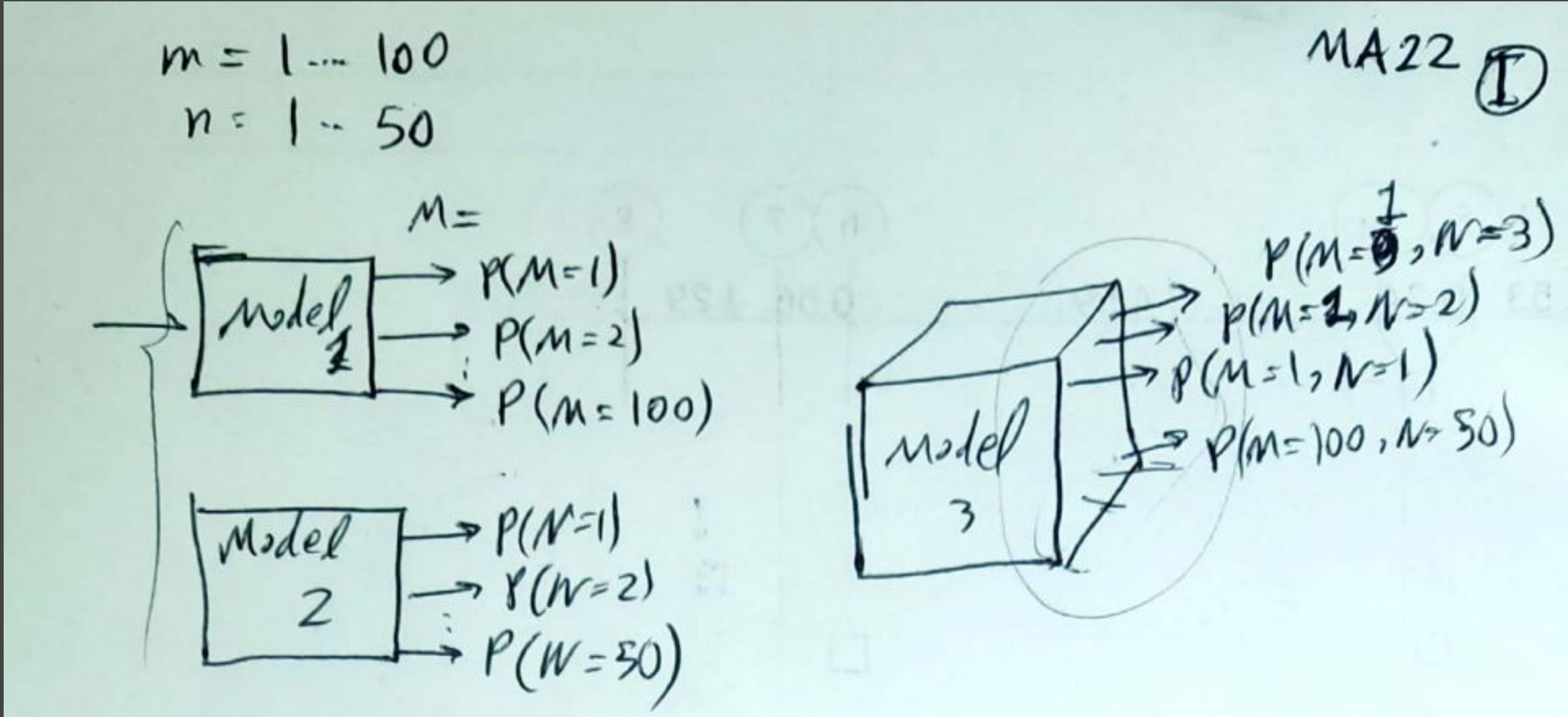
  $\Rightarrow P(M = m, N = n) = \Pr(N = n)\ \Pr(M = m)$

- Continuous case:

  $p(y \mid x) = p(y) \quad \Rightarrow \quad p(x,y) = p(x)\, p(y)$

Independence and model complexity



More than two variables

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- $p(x_1, x_2, x_3, \ldots, x_m)$
- Pairwise independence
  - Every pair of variables x_i, x_j are independent
- Mutual Independence
  - $p(x_i$ | any **subset** of other variables $) = p(x_i)$
  - $p(x_1, x_2, \ldots, x_m) = p(x_1)\, p(x_2) \ldots p(x_m)$

More than two variables

$$p(\underbrace{x_1, x_2, \ldots, x_n}_{Y}) = p(x_1 \mid x_2, x_3, \ldots x_n)\, p(x_2, x_3, \ldots, x_n)$$

$$= p(x_1 \mid x_2, \ldots\ x_n)\, p(x_2 \mid x_3 \ldots x_n)\, p(x_3 \ldots x_n)$$

$$= p(x_1 \mid x_2 \ldots x_n)\, p(x_2 \mid x_3 \ldots x_n) \cdot p(x_3 \mid x_4 \ldots x_n)$$

$$\ldots p(x_{n-1} \mid x_n)\, p(x_n)$$

Chain Rule

$x_1, x_2, \ldots x_n$ idependent:

$$p(x_1, x_2, \ldots, x_n) = p(x_1)\, p(x_2) \ldots p(x_{n-1})\, p(x_n)$$

testing independence

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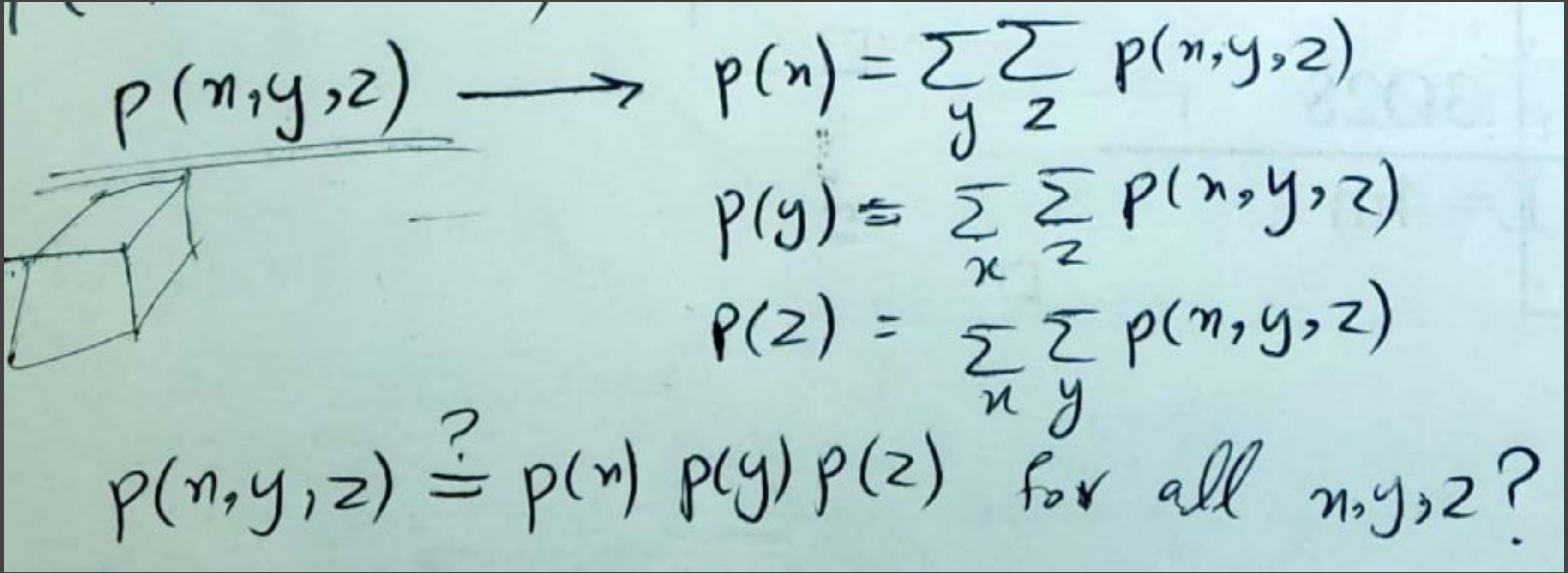
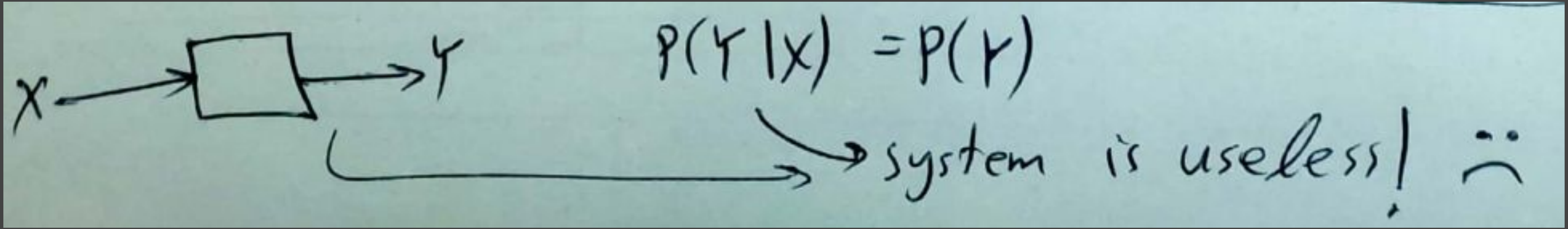


Sometimes dependence is desired



Example

- Are "Having a cloudy morning" and "getting wet" dependent?
- $P(W \mid C) = P(W)$?

Example

- Are "Having a cloudy morning" and "getting wet" dependent?
- $P(W \mid C) \neq P(W)$

Example

- Knowing that we had a rainfall
  - Are "Having a cloudy morning" and "getting wet" dependent?

Conditional Independence

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Conditional Independence

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Conditional Independence

- Knowing that we had a rainfall
  - Are "Having a cloudy morning" and "getting wet" dependent?
  - $P(W \mid R, C) = P(W \mid R)$
  - $P(W, C \mid R) = P(W \mid R)\ P(C \mid R)$
- W and C are conditionally independent given R

Conditional Independence

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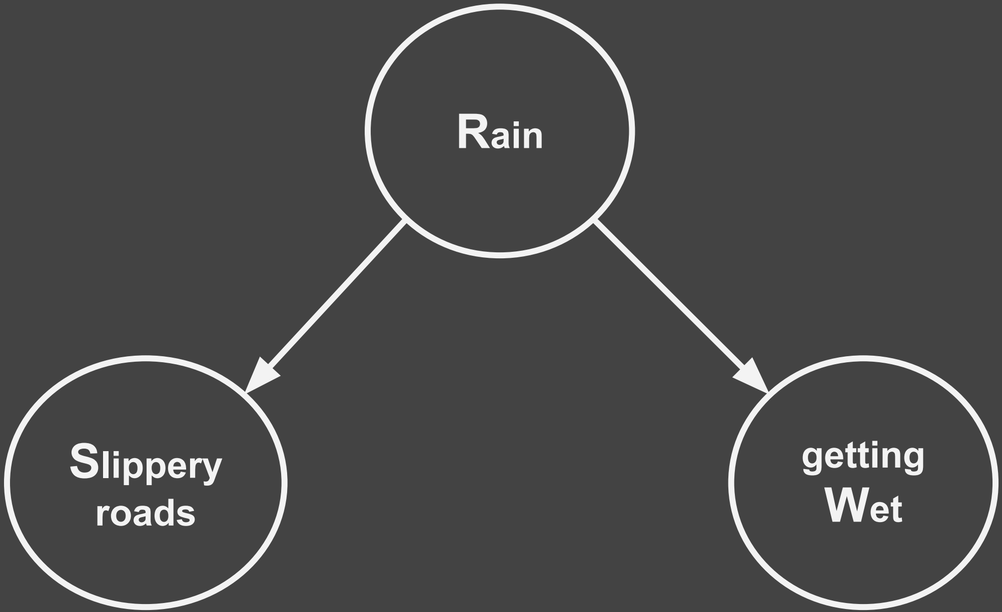
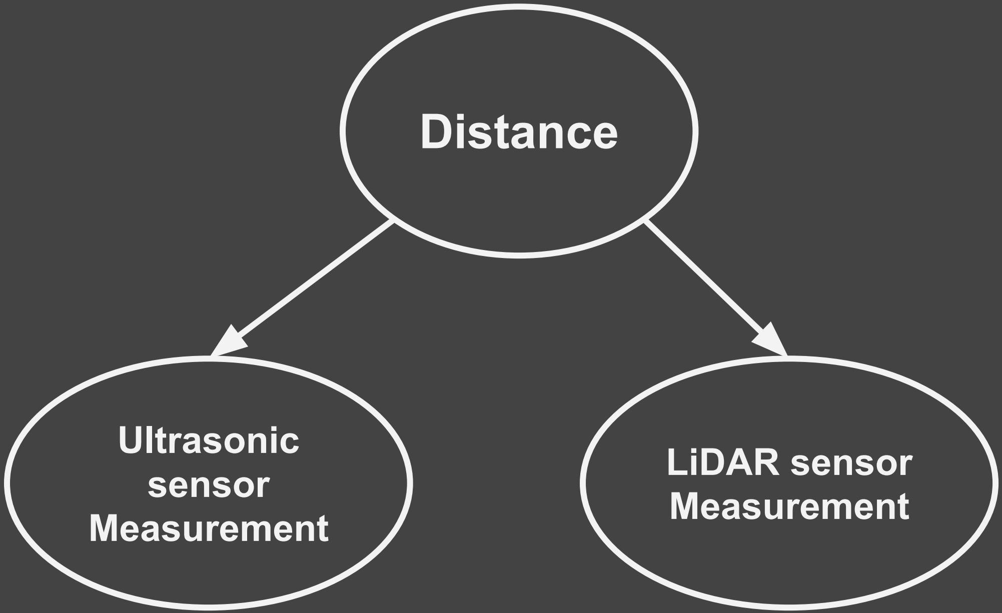
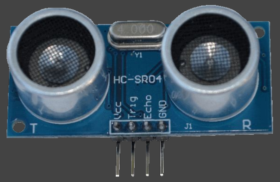


Conditional Independence





Conditioning can destroy independence

$$P(X \mid Y) = P(X)$$

Conditioning can destroy independence

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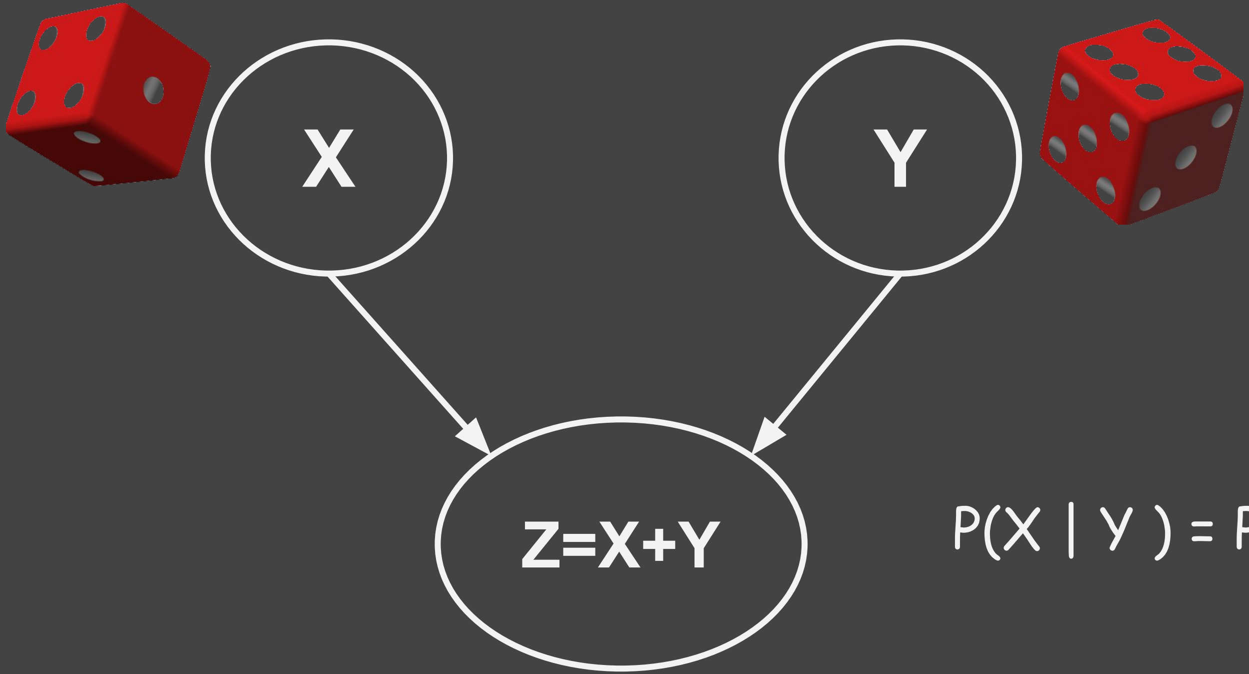

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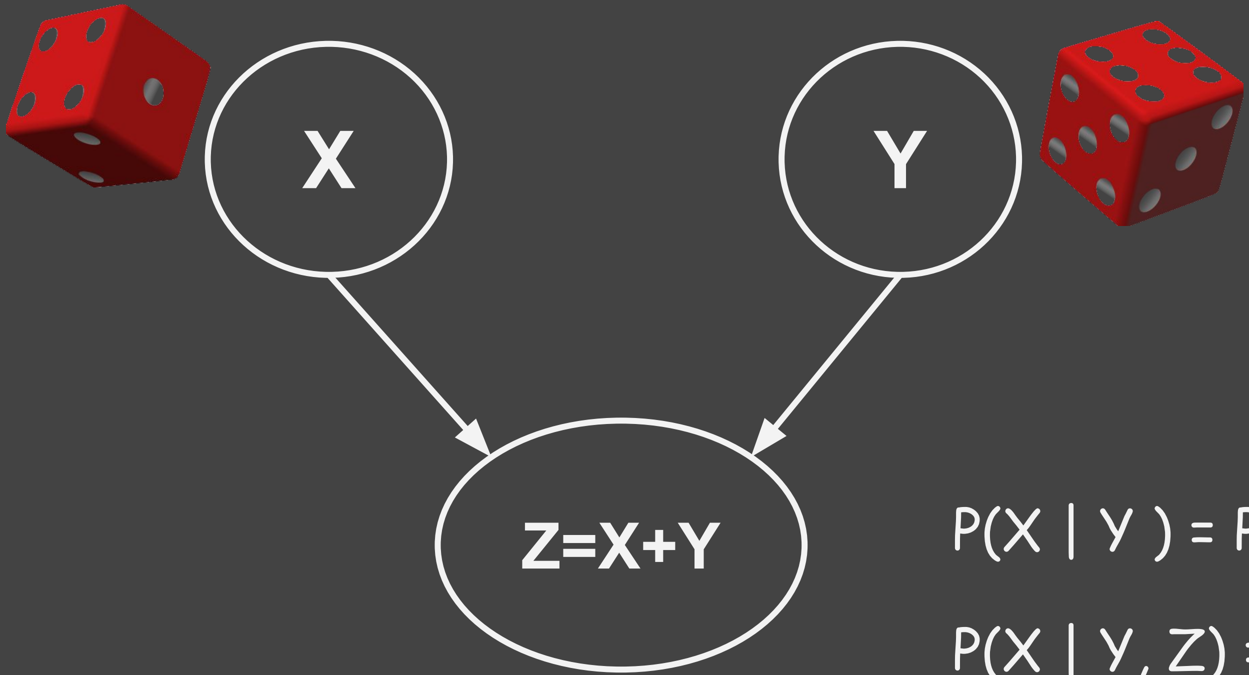
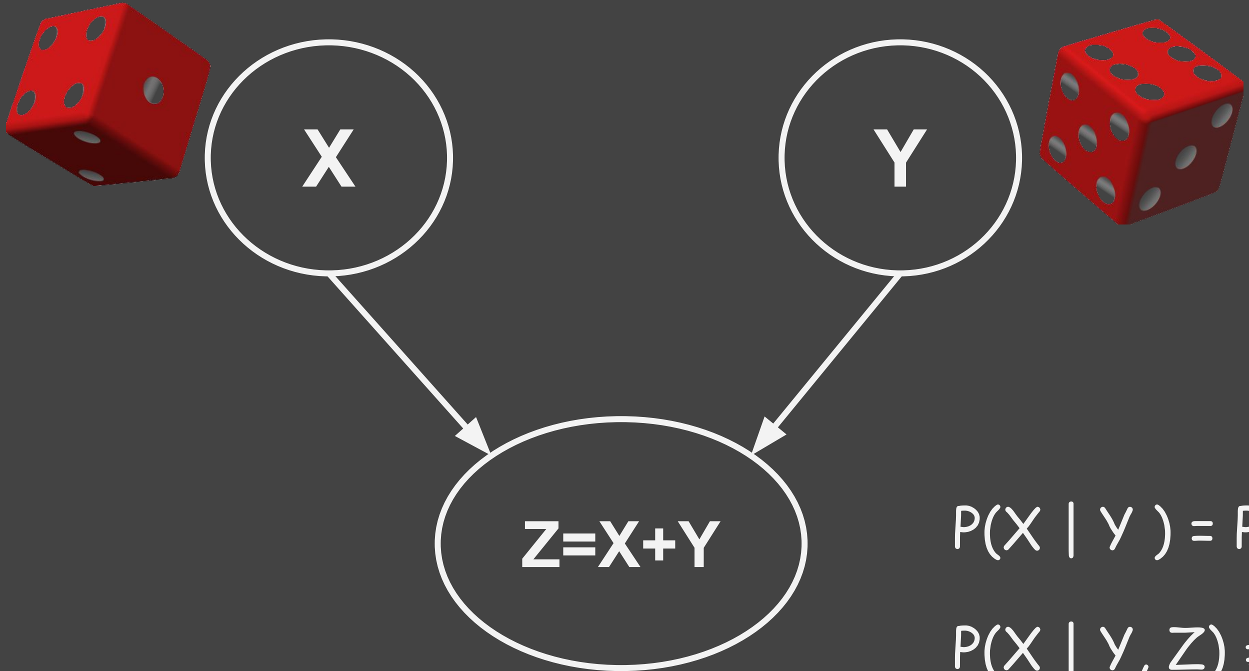


$$P(X \mid Y) = P(X)$$

$$P(X \mid Y, Z) = P(X \mid Z)\ ?$$

Conditioning can destroy independence



$$P(X \mid Y) = P(X)$$

$$P(X \mid Y, Z) \neq P(X \mid Z)$$

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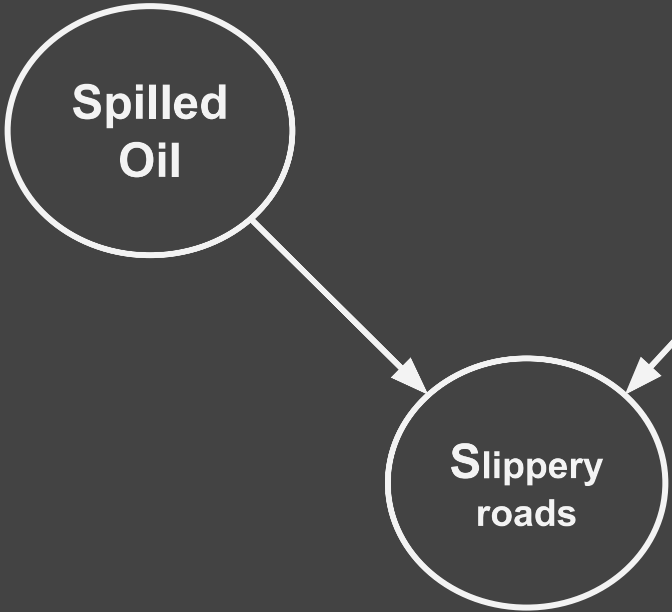
Conditioning can destroy independence



Conditional independence reduce complexity?

- Tabular representation P(C,R,W)
- General case: how many independent parameters in general?

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- Fully independent case: P(C,R,W) = P(C) P(R) P(W)
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Conditional independence reduce complexity?

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- Tabular representation P(C,R,W)
- General case: how many independent parameters in general? 7
- Fully independent case: P(C,R,W) = P(C) P(R) P(W)
  - How many parameters? 3
- Conditionally independent: P(C,R,W): P(W | R, C) = P(W | R)
  - How many parameters?

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  - How many parameters?
  - $P(C,R,W) = P(W \mid C, R)\, P(C,R) = P(W \mid R)\, P(C,R)$
  - $P(W \mid R)$:
  - $P(C,R)$:

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  - P(C,R):

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  - $P(W \mid R)$: $P(W = 0 \mid R = 0)$, $P(W = 0 \mid R = 1)$: 2 parameters
  - $P(C,R)$: 3 parameters
  - $P(C,R,W) = P(W \mid C, R)\, P(C,R)$: 5 parameters

Conditional independence reduce complexity?

- Tabular representation P(C,R,W)
- General case: 7 parameters
- Fully independent case: 3 parameters
- Conditionally independent: 5 parameters