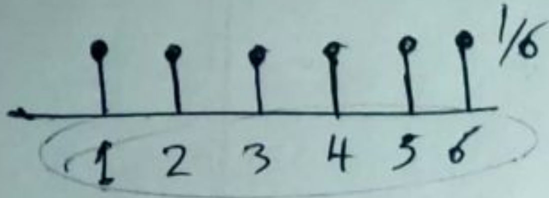
Mathematics for AI

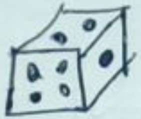
Lecture 24

Common Probability Distributions, Bernoulli distribution, Binomial distribution, Poisson Distribution, Uniform Distribution, Exponential Distribution, Gaussian Distribution, Multivariate Gaussian Distribution

Discrete Uniform Distribution



discrete uniform distribution



$x \in \{1, 2, \dots, N\}$ $\qquad p(x) = \frac{1}{N}$

$$\mu = \sum_{x=1}^{N} x \, p(x) = \sum_{x=1}^{N} x \frac{1}{N} = \frac{1}{N} \sum_{x=1}^{N} x = \frac{1}{N} \frac{N(N+1)}{2}$$

$$= \frac{N+1}{2}$$

Bernoulli distribution

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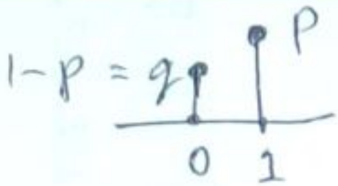
Bernoulli distribution

$x \in \{0, 1\}$

$Pr(X=1) = p$

$Pr(X=0) = q = 1-p$



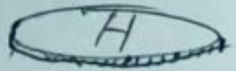
$$\mu = \sum x\, p(x) = 0\, p(0) + 1\, p(1) = 0 \cdot q + 1 \cdot p = p$$

$$\sigma^2 = \sum_x (x-\mu)^2 p(x) = \sum (x-p)^2 p(x) = (1-p)^2 p(1) + (0-p)^2 p(0)$$

$$= (1-p)^2 p + p^2 (1-p)$$

$$= p(1-p)\,[1-p+p] = p(1-p)$$

$$= pq$$



Binomial distribution

Binomial distribution MA24 II

Repeat a Bernoulli Process $\underline{n}$ times.

X: number of 1's (number of successes)

$$p(k) = \Pr(X=k) = \binom{n}{k} p^k q^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = \sum_{k=0}^{n} k \Pr(X=k) = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k} = np$$

$$\sigma^2 = npq$$

$$np(1-p)$$

Geometric distribution

- Geometric distribution

Repeat a Bernoulli process until the first sucess

X: no of Repetitions until the first 1 (inclusive) (exclusive)

inclusive: $P(X=k) = q^{k-1}p = (1-p)^{k-1}p$

exnclusive: $P(X=k) = q^k p = (1-p)^k p$

$\mu = \frac{1}{p}$

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Multinomial distribution

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Multinomial Distribution $X_1, X_2, \dots, X_L$

$P_1, \quad P_2, \quad P_n, q_n$

$p(X=i) = P_i \quad i=1,\dots,L \qquad \sum_{i=1}^{L} p_i = 1$

$$Pr(X_1=k_1, X_2=k_2, \dots, X_L=k_L) = \frac{n!}{k_1!\,k_2!\cdots k_L!}\, p_1^{k_1}\, p_2^{k_2} \cdots p_L^{k_L}$$

$$\sum k_i = n$$

Poisson Distribution

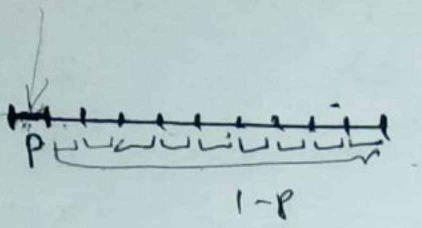
Multinomial Distribution $X_1, X_2, -, X_L$

$p(X=i) = P_i \quad i=1,\dots,L \qquad P_1, \quad P_2 \qquad P_n, q_n \qquad \sum_{i=1}^{L} p_i = 1$

$$Pr(X_1 = k_1, X_2 = k_2, \dots, X_L = k_L) = \frac{n!}{k_1!\, k_2! \dots k_L!} p_1^{k_1} p_2^{k_2} \dots p_L^{k_L}$$

$\sum k_i = n$

Poisson Distribution



Divide the interval into $\underline{\underline{n}}$ parts.
bins

$n \to \infty$

$P(X=k)$

As $n \to \infty$, probability of two events occurring in the same bin $\to 0$.

$n \to \infty, p \to 0$

$P(X=k) \simeq$ Binomial Distribution

$P(X=k) \cong \binom{n}{k} p^k (1-p)^{n-k}$ $\mu = np$

Poisson Distribution

$P(X=k) \simeq$ Binomial Distribution

$$P(X=k) \cong \binom{n}{k} p^k (1-p)^{n-k} \qquad \mu = np$$

λ: average no of occurances in the interval

1. Take a binomial distribution with $\mu = \lambda = np$

$$\Rightarrow p = \lambda/n$$

2. $n \to \infty$

$$P(X=k) = \lim_{n\to\infty} \binom{n}{k} \left(\lambda/n\right)^k \left(1-\lambda/n\right)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!}$$

Poisson Distribution: Example

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$\lambda = 200$ cars pass per minutes

$$P(X=k) = e^{-200} \frac{(200)^k}{k!}$$

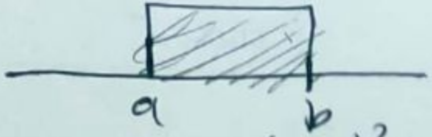
(Continuous) Uniform Distribution

Continuous Distributions:

Uniform Distribution $U(a, b)$

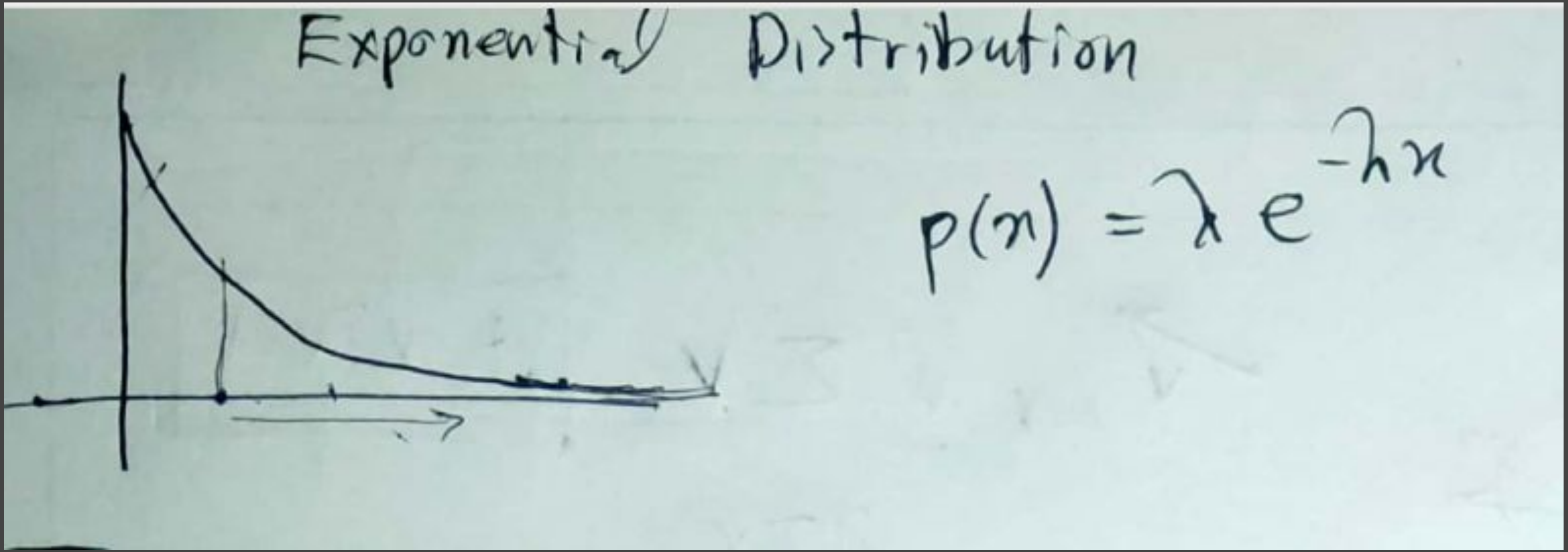
$$p(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$$

$X \sim U(a, b)$



$$\mu = \frac{b+a}{2} = \int_a^b \frac{1}{b-a}\, x \, dx = \frac{(b-a)^2/2}{b-a}$$

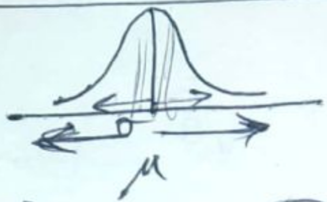
Exponential Distribution



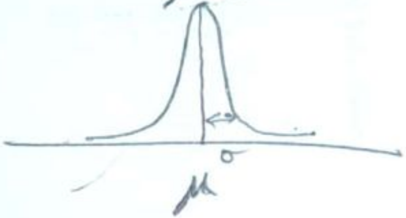
Normal (Gaussian) Distribution

Normal Distribution

Gaussian Distribution



$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



$X \sim p_X(x)$

$Y \sim p_Y(x)$

X, Y independent

Sum of independent random variables and the convolution of distributions

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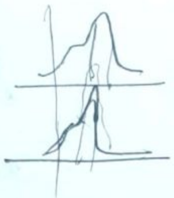
$X \sim p_X(x)$ $\quad$ X, Y independent

$Y \sim p_Y(x)$

$Z = X + Y$ $\quad$ $p_Z(z) = p_X * p_Y$

$Z' = \frac{X+Y}{2}$ $\quad$ $p_{Z'}(z') =$



$X_1, X_2, \ldots X_n$ $\quad$ n ~~random~~ independent random variables. $\quad Z = \frac{X_1 + X_2 + \cdots + X_n}{n}$

with the same distribution.

$Z' = X_1 + X_2 + \cdots + X_n$

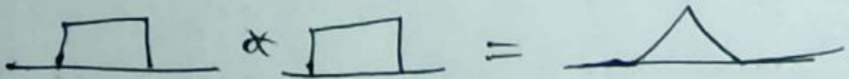
Central Limit Theorem (CLT)

$X_1, X_2, \ldots X_n$ n ~~random~~ independent random variables.

with the same distribution.

$$Z = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$Z' = X_1 + X_2 + \cdots + X_n$$



Central Limit Theorem (CLT) قضیه حد مرکزی

Joint distribution of two independent Gaussian random variables

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Random vector

MA24 (V)

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad X_1 \sim N(\mu_1, \sigma_1^2)$$

$$X_2 \sim N(\mu_2, \sigma_2^2) \qquad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

X_1, X_2 are independent

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

$$p(X) = p\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\right) = p(X_1, X_2) = p(X_1)\, p(X_2)$$

$$= \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x_2-\mu_2)^2}{\sigma_2^2}}$$

$$= \frac{1}{\sigma_1 \sigma_2 \cancel{\sqrt{\ }}\, 2\pi} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - \frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right)$$

$$= \frac{1}{\sigma_1 \sigma_2\, 2\pi} \exp\left(-\frac{1}{2} \underbrace{\begin{bmatrix} x_1-\mu_1, & x_2-\mu_2 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} \sigma_1^{-2} & 0 \\ 0 & \sigma_2^{-2} \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}}_{2\times 1}\right)$$

Towards dependent case

$$= \frac{1}{\sigma_1 \sigma_2 \, 2\pi} \exp\left(- \frac{(x_1-\mu_1)^2}{2\sigma_1^2} + \frac{(x_2-\mu_2)^2}{2\sigma_2^2} \right)$$

$$= \frac{1}{\sigma_1 \sigma_2 \, 2\pi} \exp\left(-\tfrac{1}{2} \underbrace{\begin{bmatrix} x_1-\mu_1 , & x_2-\mu_2 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} \sigma_1^{-2} & 0 \\ 0 & \sigma_2^{-2} \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}}_{2\times 1} \right)$$

$$= \frac{1}{\sqrt{\det(\Sigma)} \, \left(\sqrt{2\pi}\right)^2} \exp\left(-\tfrac{1}{2} (x-\mu)^T \, \Sigma^{-1} \, (x-\mu) \right)$$

$$(x-\mu) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

Also true for where X_1 & X_2 are dependent

Multivariate Gaussian Distribution: General Form

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$X_1, X_2, \ldots, X_n \qquad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \qquad x \in \mathbb{R}^n \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$p(\vec{x}) = \frac{1}{\sqrt{|\Sigma|}\,(\sqrt{2\pi})^n} \exp\left(-\tfrac{1}{2}(x-\mu)\,\Sigma^{-1}(x-\mu)\right)$$

v_1, v_2 Eigenvectors of Σ