Mathematics for AI

Lecture 5
Linear Equations, Singular and Non-singular matrices

* Column Rank and Row Rank

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$$\begin{bmatrix}1&2&3&0\\2&3&5&1\\3&4&7&2\end{bmatrix} = \underbrace{\begin{bmatrix}1&2\\2&3\\3&4\end{bmatrix}}_{3\times 2}\underbrace{\begin{bmatrix}1&0&1&2\\0&1&1&-1\end{bmatrix}}_{2\times 4}$$

Matrix Factorization

Row Rank = Col Rank

$$\begin{bmatrix}c_1 & c_2 & \cdots & c_n\end{bmatrix}$$

$L = [\,]$

for $i = 1 .. n$

  if $c_i \notin \overbrace{\text{span}(c_1, \ldots, c_{i-1})}^{\text{span}(L)}$

    $L.\text{append}(c_i)$

Column Rank = Row Rank

Column Rank and Row Rank

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$$\begin{array}{cc} & n \\ m & \begin{bmatrix} & A & \end{bmatrix} \end{array} = \begin{array}{cc} & r \\ m & \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} \end{array} \begin{array}{cc} & n \\ r & \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \end{array}$$

a basis for column space. a basis for row space

column rank $= r \leq \min(m,n)$

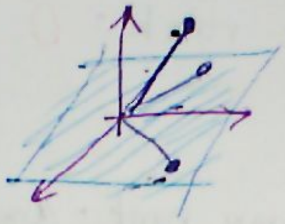
$$\boxed{\begin{aligned} \text{Row Rank}(A) &= \text{Column Rank}(A) \\ &\overset{def}{=} \text{Rank}(A) \end{aligned}}$$

رتبه.

$$\text{Rank}(A) = \dim(\text{Colspace}(A)) = \dim(\text{RowSpace}(A))$$

"Most" matrices have full rank

$$\begin{array}{cc} & 3 \\ 3 & \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \\ & A \end{array}$$

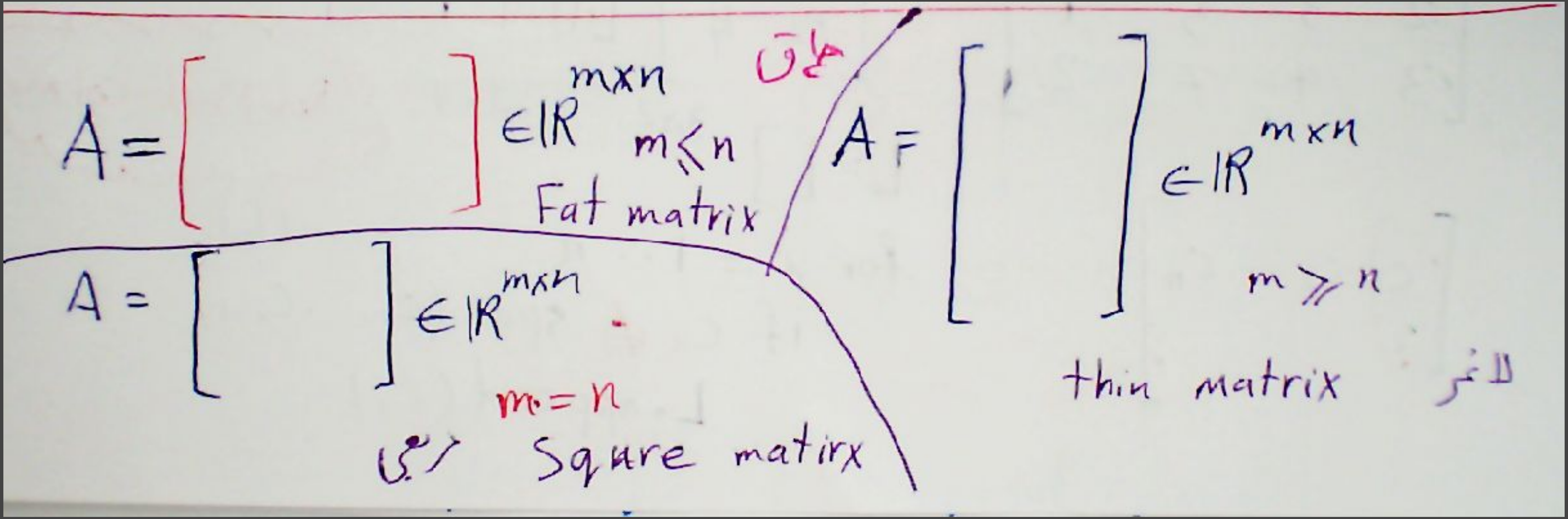


if A is a random 3×3 matrix (uniform distribution / nondegenerate normal distribution)

$\operatorname{rank}(A) = 3$ with probability $= 1$

almost Always

thin and fat matrices



full-rank and rank-deficient

V 9

$A \in \mathbb{R}^{m \times n}$ $\qquad \text{rank}(A) \leq \min(m,n)$

$$\begin{cases} \text{rank}(A) = \min(m,n) & \text{full-rank} \\ \text{rank}(A) < \min(m,n) & \text{rank-deficient} \end{cases}$$

رتبه کامل

رتبه ناقص

A is full-rank

A has full rank

full-rank and rank-deficient

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V 9

$A \in \mathbb{R}^{m \times n}$ $\qquad$ $\text{rank}(A) \leq \min(m,n)$

$$\begin{cases} \text{rank}(A) = \min(m,n) & \text{full-rank} \\ \text{rank}(A) < \min(m,n) & \text{rank-deficient} \end{cases}$$

رتبه کامل

رتبه ناقص

A is full-rank

A has full rank

$A \in \mathbb{R}^{n \times n}$

$\text{rank}(A) = n$ full-rank

$\text{rank}(A) < n$ rank-deficient

Linear Equations

Linear Equations

3 equations معادله

$$\begin{cases} x + z = 4 \\ x - y = 3 \\ x + y + z = 2 \end{cases}$$

$$1x + 0y + 1z = 4$$
$$1x + (-1)y + 0z = 3$$
$$1x + 1y + 1z = 2$$

3 unknowns x, y, z

مجهول

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

m equations

n unknowns

$$A \vec{x} = \vec{b}$$

Independent Equations

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Eq1 $2x + y + 2z = 6$

Eq2 $x - y = 3$

Eq3 $x + 2y + 2z = 3$

3 Equations

Eq1 − Eq2 = Eq3 2 Independet Equations

$$\underbrace{\begin{bmatrix} 2 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 2 \end{bmatrix}}_{\text{rank deficient}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

Let's focus on a special case

$$\underset{\mathbb{R}^{m\times n}}{A}\ \underset{\mathbb{R}^{n}}{x} = \underset{\mathbb{R}^{m}}{b}$$

m equation

n unknowns

Special case $\begin{cases} m = n \Rightarrow A \text{ square} \left(\begin{array}{l} n \text{ Equations} \\ n \text{ Unknowns} \end{array}\right) \\ A \text{ has full rank} \left(\begin{array}{l} \text{Independent} \\ \text{equations} \end{array}\right) \end{cases}$

~~rank~~

$$\equiv \begin{cases} A \in \mathbb{R}^{n\times n} \\ \operatorname{rank}(A) = n \end{cases}$$

Linear Equations - Geometric Interpretation

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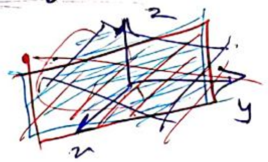
$$A \quad \vec{x} = b$$

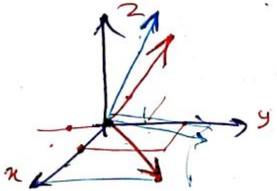
$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$$2x + y + z = 3$$

$$x - y = 3$$



$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} z = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$



$$\boxed{A\vec{x} = b}$$

Singular and Nonsingular Matrices

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$A \in \mathbb{R}^{n\times n}$ square

A <u>has full rank</u>, <u>nonsingular</u>

<u>$\text{rank}(A) = n$</u>, nondegenerate

<u>Invertible معکوس‌پذیر</u>

Nonsingular Matrices and Linear Transformations

$A \in \mathbb{R}^{n\times n}$ square

A has full rank, nonsingular

$\text{rank}(A) = n$, nondegenerate

Invertible معکوس پذیر

one-to-one

injective یک به یک

onto, surjective پوشا

$f(\vec{x}) = A\vec{x} \qquad A \in \mathbb{R}^{n\times n}$

$f: \mathbb{R}^n \to \mathbb{R}^n$

find x s.t. $f(x) = b$

یک به یک

$\boxed{x=y} \Leftarrow x-y=0 \Leftarrow d=0$

$\Uparrow$

$f(x) = f(y) \Rightarrow Ax = Ay \Rightarrow A\underbrace{(x-y)}_{d} = 0$

$$Ad = 0 \Rightarrow \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = d_1 a_1 + d_2 a_2 + \cdots + d_n a_n = 0 \Rightarrow \begin{cases} d_1 = 0 \\ d_2 = 0 \\ \vdots \\ d_n = 0 \end{cases}$$

Nonsingular Matrices and Linear Transformations

$f: \mathbb{R}^n \to \mathbb{R}^n$

$y \in \mathbb{R}^n \; \exists x \overset{?}{} \; f(x) = y$

$Ax \overset{?}{=} y$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = y$$

$a_1, \cdots, a_n$ form a basis for $\mathbb{R}^n \Rightarrow \exists \; x_1, x_2, \cdots, x_n$

$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = y$

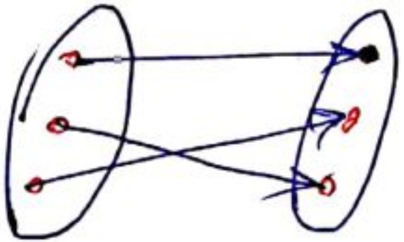
$\Rightarrow \exists x \in \mathbb{R}^n \Rightarrow Ax = f(x) = y$

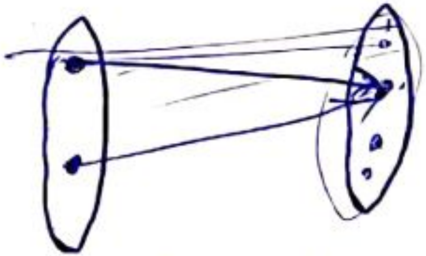
f پوشا است

Nonsingular Matrices and Linear Transformations

f is one-to-one and onto $\Rightarrow$

$\exists f^{-1}$ $\quad f^{-1}(f(x)) = x$ for all $x \in \mathbb{R}^n$





f is linear and bijective $\qquad f(x) = Ax$

f^{-1} is ? and //

Nonsingular Matrices and the Inverse Map

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f is linear and bijective $\quad f(x) = Ax$

f^{-1} is ? and //

$x_1 \doteq f^{-1}(y_1) \Rightarrow y_1 = Ax_1$

$x_2 \doteq f^{-1}(y_2) \Rightarrow y_2 = Ax_2$

$$f^{-1}(\alpha y_1 + \beta y_2)$$

$$= f^{-1}(\alpha A x_1 + \beta A x_2) = f^{-1}(A(\alpha x_1 + \beta x_2)) = f^{-1}(f(\alpha x_1 + \beta x_2))$$

$$= \alpha x_1 + \beta x_2 \Rightarrow \boxed{f^{-1}(\alpha y_1 + \beta y_2) = \alpha f^{-1}(y) + \beta f^{-1}(y)}$$

f^{-1} is linear

The Inverse Matrix

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$\because f^{-1}$ is linear

$f^{-1}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is linear $\Rightarrow \exists B \Rightarrow f^{-1}(y) = By \quad \forall y \in \mathbb{R}^n$

$B = [b_1\ b_2 \cdots b_n] \qquad b_i = f^{-1}(e_i) = f^{-1}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) \rightarrow i$

The Inverse Matrix

$$f^{-1}(f(x)) = x \Rightarrow B(Ax) = x \Rightarrow (BA)x = x$$

$$\forall x$$

$$\boxed{(AB)C = A(BC)}$$

$$\Rightarrow \cancel{(BA)} \quad (BA)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} IA &= A \\ AI &= A \end{aligned}}$$

$$(BA)I = I \Rightarrow BA = I$$

The Inverse Matrix

$$(BA)\,I = I \Rightarrow BA = I$$

$f(x) = Ax$

$f^{-1}(x) = Bx$

B is called the inverse of A and is denoted by A^{-1} $\Rightarrow$ Every nonsingular matrix has an Inverse

$$\boxed{A^{-1}A = I} \qquad \boxed{AA^{-1} = I} \qquad \boxed{\forall x \;\; AA^{-1}x = x}$$

$$(A^{-1})^{-1} = A$$

$$\begin{cases} A^{-1}A = I \Rightarrow AA^{-1} = I \Rightarrow \overbrace{BA}^{I}A^{-1} = B \Rightarrow \\ BA = I \Rightarrow \end{cases}$$

$$\Rightarrow A^{-1} = B$$

Solve linear equations using Inverse Matrix

$$Ax = y \implies \text{find } A^{-1} \implies A^{-1}Ax = A^{-1}y$$

How?

$$\implies \boxed{x = A^{-1}y}$$

np.linalg.inv(A)

Solve linear equations using Inverse Matrix

$$Ax = y \Rightarrow x = A^{-1}y$$

$$\underset{n\times n}{\overset{\checkmark}{A}}\ \underset{n\times p}{\overset{?}{X}} = \underset{n\times p}{\overset{\checkmark}{Y}} \Rightarrow X = A^{-1}Y$$

$$A\begin{bmatrix} x_1 & x_2 & \cdots & x_p \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & \cdots & y_p \end{bmatrix}$$

Null vectors of nonsingular matrices

$$\begin{bmatrix} & A & \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$\cancel{A \text{ is singular} \Rightarrow \exists\, \beta_1, \beta_2, \dots, \beta_n}$

A is nonsingular $\quad x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = 0$

$$\Rightarrow x_1 = x_2 = \cdots = x_n = 0$$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

A is nonsingular $\quad \boxed{Ax = 0 \Rightarrow x = 0}$

Null vectors of singular matrices

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A is singular $\exists\, x \neq 0$ s.t. $Ax = 0$

null vector

$\Rightarrow$ A has at least one nonzero null vetor.

Singular Matrices and linear equations

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$A \in \mathbb{R}^{n\times n}$ is singular = rank deficient $\qquad Ax=b$

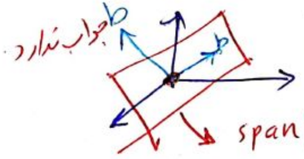
$\operatorname{rank}(a) < n$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b \Rightarrow b = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

$b \in \operatorname{span}(a_1, \dots, a_n)$

$\dim(\operatorname{span}(a_1, \dots, a_n)) < n$

column space



$$A^{-1}Ax = \overset{x}{\cancel{I}} \quad \forall x$$

$$A^{-1}\underbrace{\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} x}_{Y \in C(A)} = \cancel{I}\, x$$

columnspace

$$\cancel{A}\, A \underbrace{A^{-1} x}_{Y \in C(A} = x \quad \forall x$$

Singular Matrices and linear equations

A singular $\qquad A\vec{x}=\vec{b} \qquad$ n5 VI

$$A=\begin{bmatrix} a_1 & a_2 & - & a_n \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=\vec{b}$$

$\text{rank}(A)<n$ / $\vec{a}_1 — \vec{a}_n$ are not independent

$$\text{span}(\vec{a}_1, \vec{a}_2, \cdots \vec{a}_n) = C(A) \neq \mathbb{R}^n$$

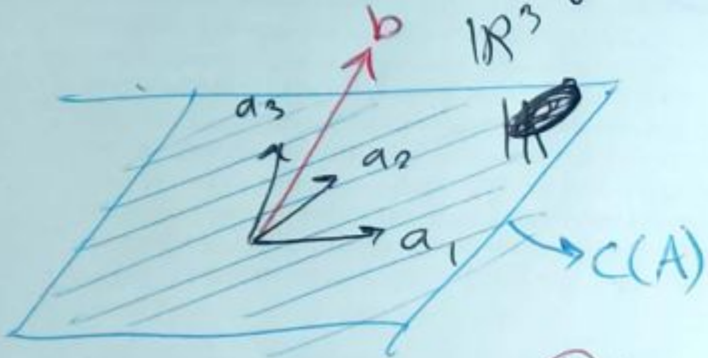
column space

$A\in\mathbb{R}^{3\times 3}$ singular $\qquad A=\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$

b $\mathbb{R}^3$

Singular Matrices and linear equations

$A \in \mathbb{R}^{3\times 3}$ singular $\quad A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$

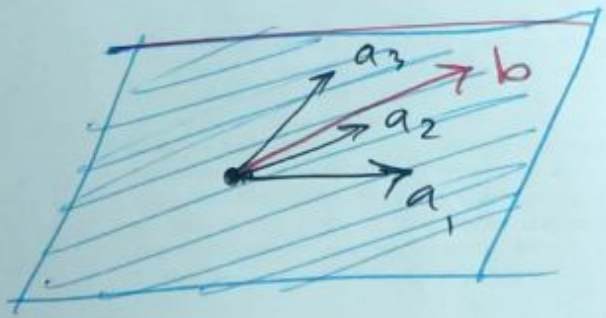


$Ax = b$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$$

$Ax = b \quad$ I $\; b \notin C(A) \Rightarrow Ax = b$ has no solution

Singular Matrices and linear equations



(II) $b \in C(A) \Rightarrow b \in \mathrm{span}([a_1, a_2, a_n])$

$\Rightarrow \exists x \quad Ax = b$

A singular $\exists v \neq 0 \quad Av = 0$

$A(x + \alpha v) = b$

$Ax + \alpha \cancel{Av}$

$Ax = b$

$\Rightarrow$ infinite no. of solution

infinitly many solutions