Mathematics for AI

Lecture 6
null space, solution to general linear equations

LU Decomposition

K. N. Toosi
University of Technology

Matrix Decomposition $\quad A = BC \quad$ $\cancel{A = B + C}$

$A = LU$

$U \rightarrow$ upper triangular

$L \rightarrow$ lower triangular

LU-decomposition

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 6 & 4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 7 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

lower $\qquad$ upper

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Permutation Matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} e & f \\ a & b \\ c & d \end{bmatrix}$$

permutation matrix

$PA \rightarrow$ permutes rows of A

$AP \rightarrow$ 〃 columns of A

$$\underline{\underline{P^{-1} = P^{T}}}$$

LU Decomposition

for any square matirix A $\exists$ P, U, V

permutation

upper tri

lower tri.

$$PA = LU$$

Solving linear equations with LU decomposition

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University of Technology

$$PA = LU$$

$$Ax = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 12 \end{bmatrix}$$

$$\Rightarrow 1 \times x = 2 \Rightarrow x = 2$$

$$4 \times x + 2y = 6 \Rightarrow 2y = -2$$

$$y = -1$$

$$z = \checkmark$$

backward substitu
substitution

$$\cancel{A x b} \; A \quad PAx = Pb$$

$$A'x = b'$$

$$L\underbrace{Ux}_{y} = b$$

$$Ly = b \Rightarrow y = \checkmark \Rightarrow Ux = y \Rightarrow x = \checkmark$$

LDU Decomposition

LDU = decomposition

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & \\ 1/2 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 6 & \\ & & 12 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Null space

A column space $C(A)$ (range A)

row space $R(A)$

null space $N(A)$ (kernel A)

$f(x)$ range$(f) = \{ f(x) \mid x \in X \}$ $f(x) = Ax$

$f : X \to Y$

kernel$(f) = \{ x \in X \mid f(x) = 0 \}$

$$N(A) = \{ x \mid Ax = 0 \}$$

فضای تهی
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null space

$A \in \mathbb{R}^{m \times n}$

$N(A)$ $x \overset{?}{\in} N(A)$

Null space

v is a null vector of $\underset{m\times n}{A}$ if $A\vec{v}=\vec{0}$

$\in \mathbb{R}^n$

the space of All null vectors of A:

$N(A) = \{ v \mid Av = 0 \}$ null space of A

فضای پوچ A

Null space is a linear subspace

$$N(A) = \{x \mid Ax = 0\}$$

null space

فضای
نول

$A \in \mathbb{R}^{m \times n}$

$$y, x \in N(A) \Rightarrow \alpha x \overset{?}{\in} N(A)$$

$$x+y \overset{?}{\in} N(A)$$

$$A(\alpha x) = \alpha A x = \alpha \cdot 0 = 0 \Rightarrow \alpha x \in N(A)$$

$$A(x+y) = Ax + Ay = 0 + 0 = 0 \Rightarrow x+y \in N(A)$$

$\Rightarrow N(A)$ is a linear subspace (of $\mathbb{R}^n$)

Column space, row space, null space, left null space

K. N. Toosi
University of Technology

$C(A)$ column space

$$\left.\begin{array}{l} \vec{x} \in C(A) \\ \vec{y} \in C(A) \end{array}\right\} \begin{array}{ll} \exists b & Ab = \vec{x} \\ \exists p & Ap = \vec{X} \end{array}$$

$$\underline{\alpha \vec{x} + \beta y} = \alpha(Ab) + \beta(Ap) = A(\alpha b + \beta b) = \underline{Aq}$$

$$\Rightarrow \alpha \vec{x} + \beta y \in C(A)$$

$C(A^T)$ row space $R(A)$

$N(A)$ $\quad \vec{x}, \vec{y} \in N(A) \Rightarrow \begin{cases} Ax = 0 \\ Ay = 0 \end{cases}$

$$A(\alpha \vec{x} + \beta \vec{y}) = \alpha A\vec{x} + \beta A\vec{y} = \alpha \vec{0} + \beta \vec{0} = 0$$

$\Rightarrow N(A)$ is a linear subspace

$$\{u \mid u^T A = 0^T\} = \{u \mid A^T u = 0\} = N(A^T)$$

$\downarrow$

left nullspace

$Ax = 0$

$Ax = 0$ $\quad 0 \in N(A)$ $\quad \boxed{\alpha H}$

$N(A)$ includer all answers to $\boxed{Ax = 0}$

General Ax=b

$Ax=b \quad A\in\mathbb{R}^{m\times n} \quad x\in\mathbb{R}^{n} \quad b\in\mathbb{R}^{m}$ (VII)

$\mathrm{rank}(A)=r\leq\min(m,n)$

$N(A)=\checkmark$

$$\begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=b \Rightarrow x_1\vec{a}_1+x_2\vec{a}_2+\cdots+x_n\vec{a}_n=b$$

$b\in\mathrm{span}(a_1,a_2,\dots,a_n)=C(A)$

$b\notin C(A)$ $\quad Ax=b$ has no ~~answers~~ solutions

$b\in C(A)\Rightarrow \exists\, x_1,x_2,\dots,x_n$ s.t.

$$x_1\vec{a}_1+x_2\vec{a}_2+\cdots+x_n\vec{a}_n=\vec{b}$$

$A\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=b \Rightarrow$ A has at least one solution!

$\exists\vec{x}: \quad A\underline{\vec{x}}=\vec{b} \qquad \vec{n}\in N(A) \quad A\vec{n}=0$

Solutions to General Linear Equations

K. N. Toosi
University of Technology

$$\underline{Ax = b}$$

$\cancel{A: m\times n}$ $\quad A \in \mathbb{R}^{m\times n}$

$\cancel{b\in}$ $\quad b \in \mathbb{R}^{m}$

$x \in \mathbb{R}^{n}$

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$b \notin C(A)$ No solution ?

$b \in C(A)$ At least one solution (x_p)

$\cancel{Ax_p = b}$ $\quad A x_p = b$

$A x_n = 0 \qquad x_n \in N(A)$

$$A(x_p + x_n) = A x_p + A x_n = b + 0 = b$$

$$A(x_p + \underbrace{\alpha x_n}_{\in N(A)}) = b$$

$$\begin{cases} A x_p = b \\ A x = b \end{cases} \Rightarrow A(x - x_p) = 0 \Rightarrow \overbrace{x - x_p}^{x_n} \in N(A)$$

$$x = x_p + x_n \qquad x_n \in N(A)$$

Solutions to General Linear Equations

$Ax = b$

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$b \notin C(A)$ No Solution!

$b \notin C(A)$ at least one solution!

1. Find x_p s.t. $A x_p = 0$
2. Find $N(A)$ (a basis for $N(A)$)

all solutions to $Ax=b$ are

$$\{ x_p + x_n \mid x_n \in N(A) \}$$

a) How to check if $b \in C(A)$?

b) How to find x_p?

Set of solutions is not a linear subspace

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$\boxed{A: m \times n}$

$$S = \{\underline{x} \mid Ax = b\} \subseteq \mathbb{R}^n$$

Does S form a linear subspace of $\mathbb{R}^n$?

$b = 0 \Rightarrow$ YES $\quad x = N(A)$

$b \neq 0 \quad Ax = b \quad A(\alpha x) = \alpha b \neq b \Rightarrow \alpha x \notin S$

$\boxed{\text{let } \alpha \neq 1}$

$x, y \in S \quad \left.\begin{matrix} Ax = b \\ Ay = b \end{matrix}\right\} \quad A(x+y) = Ax + Ay = 2b \neq b \quad x + y \notin S$

$x, y \in S \quad A(\alpha x + \beta y) = \underline{(\alpha + \beta)\, b \neq b}$ in general

S is not a linear subspace

Set of solutions is not a linear subspace

The set of solutions to $Ax=b$ III

for $b \neq 0$ is not a linear subspace.

$$x, y \in S = \{x \mid Ax = b\}$$

$$A(\alpha x + \beta y) = (\alpha + \beta) b$$

what linear combinations of x and y are in S?

$\alpha x + \beta y \in S$ if and only if $\alpha + \beta = 1$

$\Longrightarrow$ $\alpha x + \beta y$ is an affine combination

Affine subspace

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University of Technology

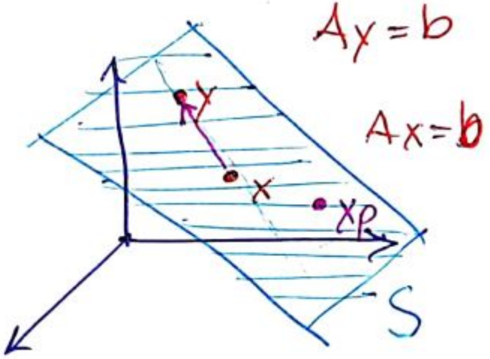
$\Rightarrow \alpha x + \beta y$ is an affine combination

$$y - x \in N(A)$$

$$N(A) = \{ x - x_p \mid Ax = b \}$$

is a linear subspace

$$D(S) = D(N(A))$$



S is an affine subspace (of $\mathbb{R}^n$)

Full column-rank case

$A \in \mathbb{R}^{m \times n}$ A has full column rank IV

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$\text{rank}(A) = r = n$

($\Rightarrow$ A has full-~~co~~rank)

$\Rightarrow \cancel{m \geq n}$ $r = n \Rightarrow m \geq n \Rightarrow A = \begin{bmatrix} \vert & \vert & \vert & \vert \\ \vert & \vert & \vert & \vert \end{bmatrix}$

$\Rightarrow$ A is a tall matrix

(thin, skinny)

$r = n \Rightarrow$ A has n independent columns

$Ax = b$ $N(A)$ $Ax = 0 \Rightarrow \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$

$\Rightarrow x = 0$ $\Rightarrow N(A) = \{0\}$

$b \notin C(A) \Rightarrow$ No Solution

$b \notin C(A) \Rightarrow$ A unique solution!

Full row-rank case

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$Ax = b$ $\qquad A \in \mathbb{R}^{m \times n}$ has full row rank $\quad$ V 15

$r = \text{rank}(A) = m$

$\Rightarrow m \leqslant n$ $\qquad$ ($\Rightarrow$ A has full rank

$A = \begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \end{bmatrix}$

A is a fat matrix
wide

$Ax = b$ $\qquad \text{rank}(A) = m$

$\Rightarrow$ A has $\underline{\underline{m}}$ idependent column

$C(A) = \mathbb{R}^m$

$\Rightarrow$ b is always in $C(A)$

$\Rightarrow$ there is Always a solution.

$\dim(N(A)) = n - r = n - m$

$x = x_p + x_n \qquad x_n \in N(A)$

Non-singular case

$A \in \mathbb{R}^{n \times n}$ is non-singular

$$\left\{\begin{array}{l} A \text{ has full column rank} \Rightarrow \text{No or unique solution} \\ A \text{ has full row rank} \Rightarrow \text{has a solution} \end{array}\right\} \Rightarrow$$

$\Rightarrow$ has a unique solution!

Rank-deficient case

$Ax = b \qquad A \in \mathbb{R}^{m \times n}$

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A is rank-deficient $\qquad r < \min(m,n)$

n

$b \notin C(A)$ No Solution

$b \in C(A)$

$\dim(N(A)) = n - r \geq 1$

$b \in C(A) \Rightarrow$ infinite solutions