Mathematics for AI

Lecture 7
Homogeneous Equations, Vector Norm, Orthogonality, Orthogonal Projection

Homogeneous Equations

$$2x + 3y - z = 0$$

$$x - 2y + 4z = 0$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2\alpha \\ -\alpha \\ \alpha \end{bmatrix} \qquad \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

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In ~~eq~~ (2) we see

In \eqref{eq:linear}

$$A\vec{x} = 0$$

$$A(\alpha\vec{x}) = \alpha A\vec{x} = \alpha\vec{0} = \vec{0}$$

$$\vec{x} \qquad \alpha\vec{x}$$

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Homogeneous Equations

Vector lenght

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$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\|\vec{x}\|^2 = x_1^2 + x_2^2 + x_3^2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{x} \cdot \vec{x}$$

$$\langle \vec{x}, \vec{x} \rangle = x^T x$$

$$\|\vec{0}\| = 0$$

Vector norm and normed vector spaces

Nonnegative: $\| x \| \geq 0$

Positive definite $\| x \| = 0 \Leftrightarrow x = 0$

Absolute homogeneity: $\| a x \| = |a| \, \| x \|$

Triangle inequality: $\| x + y \| \leq \| x \| + \| y \|$

https://en.wikipedia.org/wiki/Normed_vector_space

Example

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\|X\|_2 = \sqrt{\langle x, x \rangle} \rightarrow L_2\text{-norm}$$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n| \rightarrow L_1\text{-norm}$$

$$\|x\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + \cdots + |x_n|^p} \quad L_p\text{-norm}$$

$$L_\infty\text{-norm} \quad \|x\|_\infty = \max_i |x_i|$$

Orthogonality

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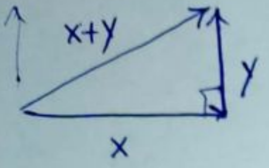
lenght $x = \sqrt{x \cdot x} = \sqrt{x^T x}$

$x, y \in V$



$x \perp y \Longrightarrow$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \Rightarrow$

$\langle x+y, x+y \rangle = \langle x, x \rangle + \langle y, y \rangle \Rightarrow$

$\langle x, x+y \rangle + \langle y, x+y \rangle = \langle x, x \rangle + \langle y, y \rangle \Rightarrow$

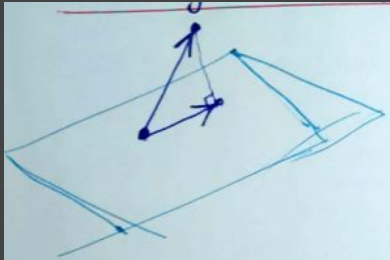
$\cancel{\langle x, x \rangle} + \langle x, y \rangle + \langle y, x \rangle + \cancel{\langle y, y \rangle} = \cancel{\langle x, x \rangle} + \cancel{\langle y, y \rangle} \Rightarrow$

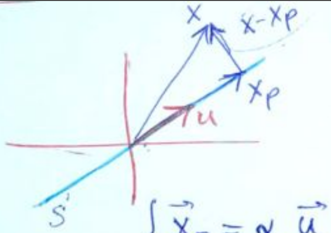
$2 \langle x, y \rangle = 0 \Rightarrow \langle x, y \rangle = 0$

$\vec{x} \perp \vec{y} \Leftrightarrow \langle x, y \rangle = 0$

$x, y \in \mathbb{R}^n \Longrightarrow x \perp y \Rightarrow x^T y = 0 = \langle x, y \rangle = x \cdot y$

Orthogonal Projection - 1D case





$$\begin{cases} \vec{x}_p = \alpha \vec{u} & \text{I} \\ u^T(x - x_p) = 0 & \text{II} \end{cases}$$

$$\text{II} \rightarrow u^T x - u^T x_p = 0 \Rightarrow u^T x = u^T x_p \xRightarrow{\text{I}} u^T x = u^T(\alpha u)$$

$$\Rightarrow (u^T x) = \alpha (u^T u) \Rightarrow \alpha = \frac{u^T x}{u^T u} \Rightarrow \underset{1\times 1}{x_p} = \alpha \underset{n\times 1}{\vec{u}} = \left(\frac{u^T x}{u^T u}\right)\vec{u}$$

$$P_S(x) = P_u(x) = \left(\frac{u^T x}{u^T u}\right)\vec{u} = \frac{1}{u^T u}\underbrace{(u^T x)}_{\text{scalar}}\underbrace{\vec{u}}_{\text{vector}} = \frac{1}{u^T u}\, u(u^T x)$$

$$= \frac{1}{u^T u}(u u^T) x$$

$$= \frac{u u^T}{u^T u}\, x$$

$$\boxed{\underset{1\times 1}{\alpha}\, \underset{n\times 1}{\vec{V}} = \underset{n\times 1}{\vec{V}}\, \underset{1\times 1}{\alpha}}$$

$$\alpha = u^T w$$

$$\alpha V = (u^T w) V \neq \underset{1\times m}{u^T}(wv)$$

$$\underset{n\times 1}{V} \underset{1\times 1}{\alpha} = \underset{n\times 1}{V} \underset{1\times 1}{(u^T w)} = \underset{n\times m}{(V u^T)} \underset{m\times 1}{w}$$

Matrix multlipicaltion of a vector and a scalar

$\alpha \vec{V} \to$ vector

$\alpha \in \mathbb{R} \quad V \in \mathbb{R}^n$

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$$\underset{1\times1}{\alpha}\ \underset{n\times1}{\vec{V}} = \underset{n\times1}{\vec{V}}\ \underset{1\times1}{\alpha}$$

compatible with matrix multiplication

$\alpha = u^T V \in \mathbb{R}$

$\alpha = u^T A v \ \mathbb{R}$

$$(u^T v) V \neq u^T v V$$

cannot multiply

$$= V (u^T v) = \underbrace{\underbrace{\underset{n\times1}{V}\ \underset{1\times n}{u^T}}_{n\times n}\ \underset{n\times1}{V}}_{n\times1}$$

$$\alpha \vec{V} = (u^T v) v = v (u^T v) = (v u^T) v$$

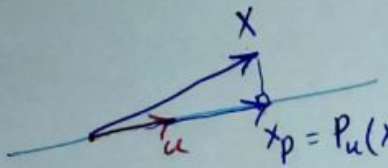
$$= v (v^T u) = v v^T u$$

$$= \underbrace{(v v^T)}_{n\times n} u$$

Rank=1

$\vec{b} \quad \vec{a}$

The projection matrix



$$x_p = P_u(x) = \frac{u^T x}{u^T u}\, u = \frac{1}{u^T u}\, u\,(u^T x) = \frac{1}{u^T u}\,\overbrace{(\underset{n\times 1}{u}\ \underset{1\times n}{u^T})}^{n\times n}\ \underset{n\times 1}{x}$$

$$P_u(x) = \underset{\in \mathbb{R}^{n\times n}}{P}\, x \qquad P = \frac{\overbrace{u u^T}^{\mathbb{R}^{n\times n}}}{\underset{\in \mathbb{R}}{u^T u}}$$

projection matrix

$$\boxed{P = \frac{1}{u^T u}\, u u^T} \qquad P^T = \frac{1}{u^T u}\,(u u^T)^T = \frac{1}{u^T u}\, u u^T \quad \text{symmetric}$$

$$P_u(x_p) = x_p$$

$$\ldots(P_u(x)) = P_u(x) \quad \forall x$$

Properties of the projection matrix

$$P_a = \frac{aa^T}{a^Ta} = \frac{aa^T}{\|a\|^2} = \left(\frac{a}{\|a\|}\right)\overbrace{\left(\frac{a}{\|a\|}\right)}^{\bar{a}}{}^T$$

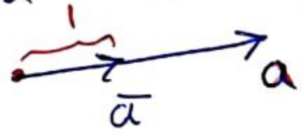
$$\bar{a} = \frac{a}{\|a\|} = \frac{a}{\sqrt{a^Ta}}$$

$\|\bar{a}\| = 1$

↓

$\bar{a}$ is a unit vector

$$P_a = \bar{a}\,\bar{a}^T$$



$$P_a^T = \frac{aa^T}{\|a\|^2} = \frac{1}{\|a\|^2}(aa^T)^T = \frac{aa^T}{\|a\|^2} = P_a \quad \text{Symmetric}$$

$$\cancel{P_a P_a x} \; = $$

$$P_a \underbrace{P_a b}_{= \alpha \vec{\bar{a}}} = P_a b \quad \forall b \Rightarrow P_a P_a = P_a$$

Properties of the Projection Matrix

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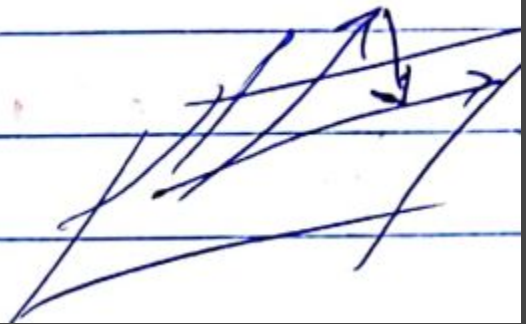
$$f_a(b) = f_a(f_a(b)) \quad \text{idempotent}$$

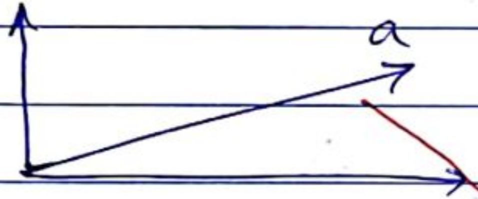
$$P_a = P_a P_a$$

$$\text{projection matrix} \begin{cases} P\,P = P \\ P^T = P \end{cases}$$



Rank of the 1-D projection matrix

$$C(P_a) = C\left(\frac{a\,a^T}{\|a\|^2}\right) = C(a) = \{\alpha \vec{a} \mid \alpha \in R\}$$

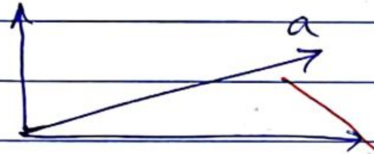


$$\frac{a\,a^T}{\|a\|^2} = \frac{\begin{bmatrix} a \end{bmatrix} [a_1\ a_2 \cdots a_n]}{\|a\|^2}$$

$$\left[\ \frac{a_1}{\|a\|^2}\, a \quad \frac{a_2}{\|a\|^2}\, a \quad \cdots \quad \beta\, a\ \right]$$

$$\mathrm{Rank}(P) = 1 = \dim(S)$$

Length of the projected vector

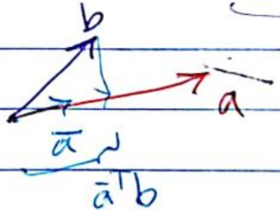


$$\frac{a a^T}{\|a\|^2} = \frac{\begin{bmatrix} a \end{bmatrix} [a_1\ a_2 \cdots a_n]}{\|a\|^2}$$

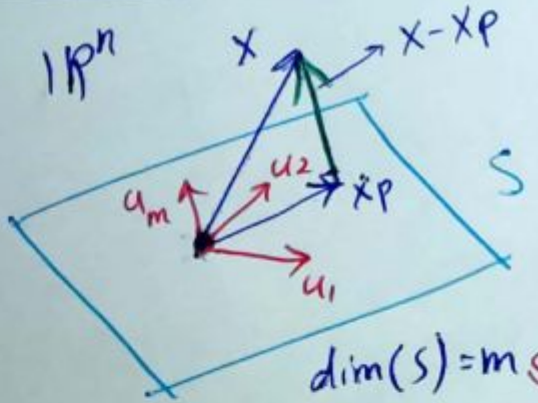
$$\begin{bmatrix} \frac{a_1}{\|a\|^2}\, a & \frac{a_2}{\|a\|^2}\, a & \cdots & a \end{bmatrix}$$

Rank(P) = 1 = dim(S)

$$f_a(b) = (\bar{a}\, \bar{a}^T)\, b = \bar{a}\,(\bar{a}^T b) = \bar{a}\ \underbrace{\langle \bar{a}, b \rangle}$$



General Projection into a linear subspace



$\dim(S) = m \leqslant n$

S: a linear subspace

$u_1, u_2, \dots, u_m$ form a basis for S

$x_p \perp x - x_p$

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$x - x_p \perp y \quad y \in S$

$x - x_p \perp u_i \quad i = 1 \dots m$

$$x_p \in \mathrm{span}(u_1 - u_m) \Rightarrow x_p = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \dots + a_m \vec{u}_m = \underbrace{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_m \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}}_{\vec{a}}$$

$$\underset{\in \mathbb{R}^n}{x_p} = \underset{n \times m}{U} \underset{\in \mathbb{R}^m}{\vec{a}}$$

$\begin{bmatrix} \; \end{bmatrix}$

General Projection into a linear subspace

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M7 III

$$(x-x_p)^T u_i = 0 \Rightarrow u_i^T x = u_i^T x_p$$

$$x_p = U\vec{a} \Rightarrow U^T \vec{x}_p = \underbrace{\underbrace{U^T}_{m\times n}\underbrace{U}_{n\times m}}_{m\times m}\vec{a}$$

$$U^T x_p = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_m^T \end{bmatrix} x_p = \begin{bmatrix} u_1^T x_p \\ u_2^T x_p \\ \vdots \\ u_m^T x_p \end{bmatrix} = \begin{bmatrix} u_1^T x \\ u_2^T x \\ \vdots \\ u_m^T x \end{bmatrix} = U^T x$$

$$\Rightarrow U^T x = U^T U a$$

$U^T U$ non-singular

$$U^T U y = 0 \Rightarrow y = 0$$

$$\underbrace{a}_{m\times 1} = \underbrace{(\underbrace{U^T U}_{m\times m})^{-1}\underbrace{U^T}_{m\times n}\underbrace{x}_{n\times 1}}$$

$$x_p = U\vec{a} = \boxed{U(U^T U)^{-1} U^T} x$$

projection matrix

$P = P^T$

$PP = P$