Mathematics for AI

Lecture 9
Determinant, Intro to Eigenvalues and Eigenvectors

Rank of product of matrices

MA9 (I)

$C(AB) \subseteq C(A)$

$I \subseteq J$

$x \in I \Rightarrow\Rightarrow x \in J$

$x \in C(AB) \Rightarrow x = (AB)z \Rightarrow x = A(\underbrace{Bz}_{y}) \Rightarrow \overset{x}{\cancel{A}} = Ay \Rightarrow x \in C(A)$

$C(ABCDE) \subseteq C(A)$

$\underset{\dim(C(AB))}{\overset{\parallel}{\mathrm{Rank}(AB)}} \leqslant \underset{\dim(C(A))}{\overset{\downarrow}{\mathrm{Rank}(A)}}$

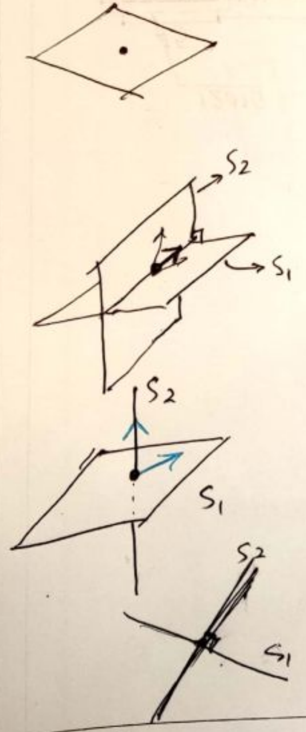
$\mathrm{Rank}(A_1 A_2 \cdots A_n) \leqslant \min($

$\mathrm{Rank}(A_1 A_2 A_3 \cdots A_n) \leqslant \min(\mathrm{Rank}(A_1), \ldots, \mathrm{Rank}(A_n))$

$\mathrm{Rank}\left(\prod_{i=1}^{n} A_i\right) \leqslant \min_i \mathrm{Rank}(A_i)$

Orthogonal subspaces

S_1, S_2 are linear subspaces



$S_1 \perp S_2 \iff \forall x \in S_1, y \in S_2$

$x_1 \perp x_2$

$S_1 \not\perp S_2$

$S_1 \perp S_2 \quad x \perp y \quad x^T y = 0$

$A \begin{cases} C(A) \\ C(A^T) = \underline{R}(A) \\ N(A) \\ N(A^T) \text{ left null spae} \end{cases}$

$C(A^T) \perp N(A)$

Row Space and Null space

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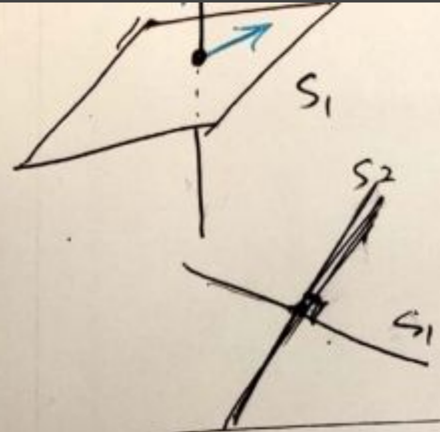
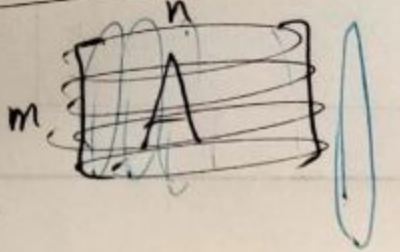


$$A \begin{cases} C(A) \\ C(A^T) = \underline{R}(A) \\ N(A) \\ N(A^T) \quad \text{left null space} \end{cases}$$

$\cancel{R(A)}$ $\quad \underline{C(A^T)} \perp \underline{N(A)}$

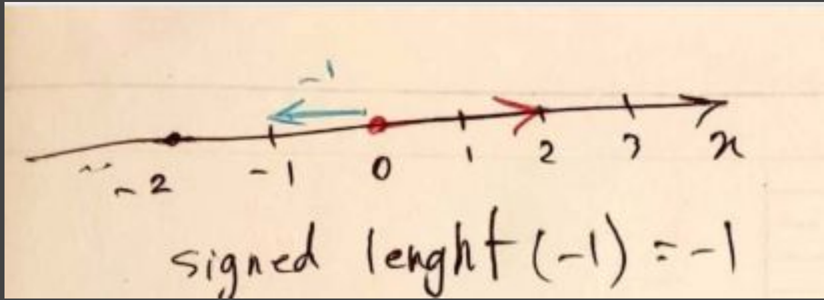


$x \in C(A^T) \Rightarrow x = A^T z \qquad C(A) \perp N(A^T)$

$y \in N(A)$

$$\langle x, y \rangle = x^T y = (A^T z)^T y = z^T A y = z^T \vec{0} = 0$$

Signed Length



Signed Area

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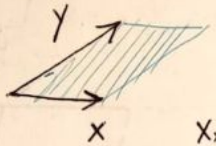
Area مساحت



$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $Y \in \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

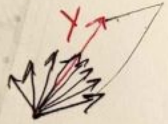
$X, Y \in \mathbb{R}^2$ $A(X,Y) = |x_1 y_2 - x_2 y_1|$

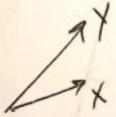
signed Area $A(X,Y) = x_1 y_2 - x_2 y_1$

$A(Y,X) = -A(X,Y) =$

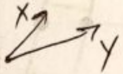
$[X \ Y] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$

$Y = aX \Rightarrow A(X,Y) = 0$



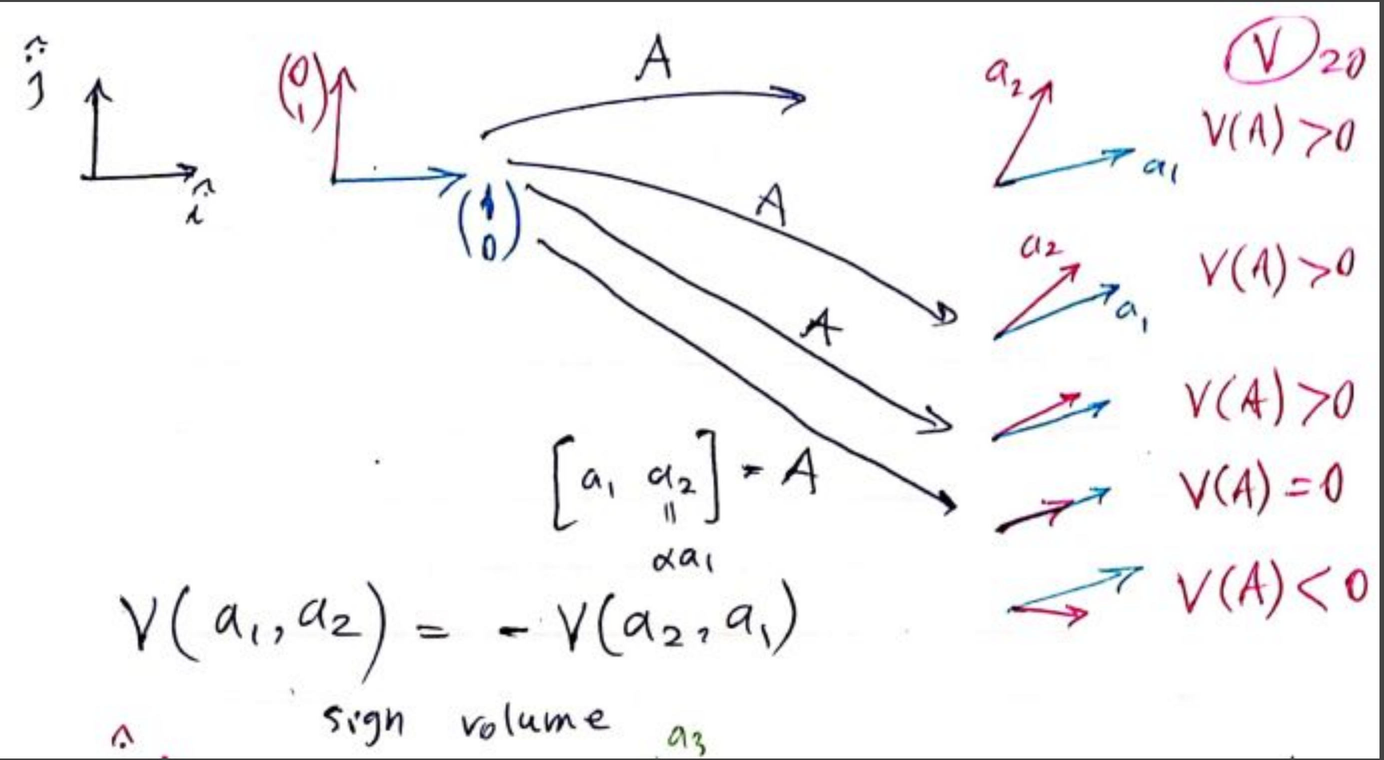


$A(X,Y) > 0$



$A(X,Y) < 0$

Signed Area



A
A
A
$\begin{pmatrix}0\\1\end{pmatrix}$
$\begin{pmatrix}1\\0\end{pmatrix}$
a_2 a_1
(V) ≥ 0
V(A) > 0
V(A) > 0
V(A) > 0
V(A) = 0
V(A) < 0
$[a_1 \ a_2] = A$
$a_2 = \alpha a_1$
$V(a_1, a_2) = -V(a_2, a_1)$
sign volume
a_3

Bilinearity of Signed Area

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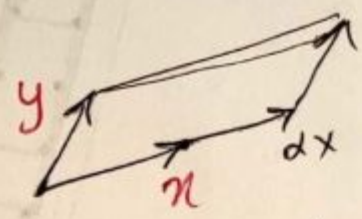
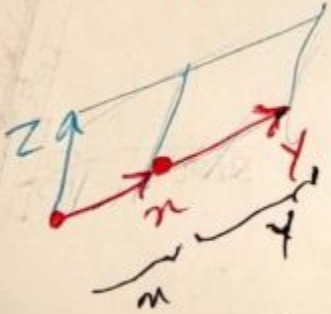


$$A(\underline{\alpha x}, y) = \alpha A(x, y)$$



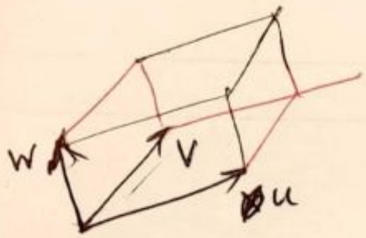
$$A(x+y, z) = A(x, z) + A(y, z)$$

$A(x, y)$ is bilinear

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

Signed Volume

MA3 III



$u, v, w \in \mathbb{R}^3$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\begin{bmatrix} u & v & w \end{bmatrix} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

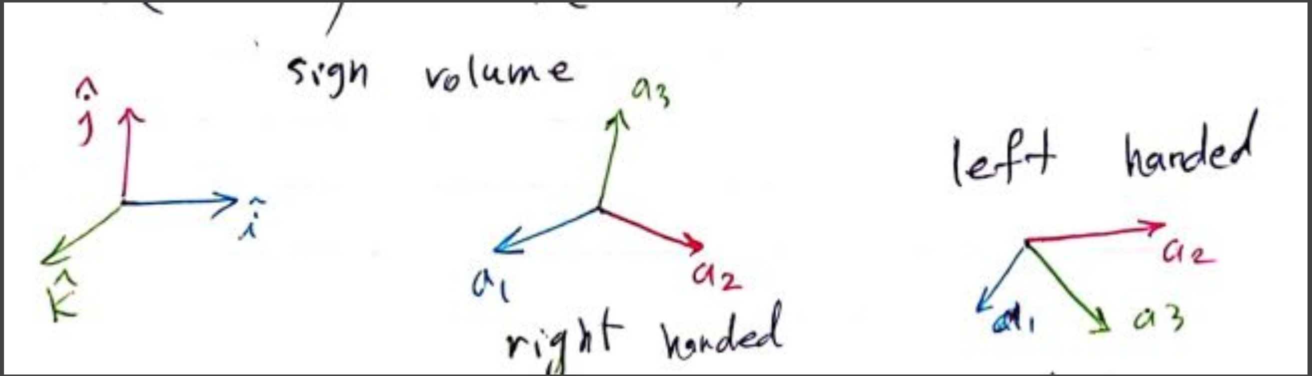
3×3

signed volume $V(u, v, w)$

$V(u, v, w)$ tri linear

$$V(\alpha \vec{u}_1 + \beta \vec{u}_2, \vec{v}, w) = \alpha V(\vec{u}_1, \vec{v}, \vec{w}) + \beta V(\vec{u}_2, \vec{v}, \vec{w})$$

Signed Volume



Trilinearlity of signed volume

signed volume $V(u, v, w)$

$V(u, v, w)$ tri linear

$$V(\alpha \vec{u}_1 + \beta \vec{u}_2, \vec{v}, w) = \alpha V(\vec{u}_1, \vec{v}, \vec{w}) + \beta V(\vec{u}_2, \vec{v}, \vec{w})$$

Determinant

$$V(a_1, a_2, a_n) = \det(A)$$

$$V(A) = \det(A) = \text{determinant of } A$$

Determinant

$u_1, u_2, \ldots, u_n \in \mathbb{R}^n$

signed hyper-volume $V(u_1, u_2, \ldots, u_n)$

$[u_1 \; u_2 \ldots u_n] \in \mathbb{R}^{n \times n}$ $\quad \det([u_1 \; u_2 \ldots u_n])$

$A = [\vec{a}_1 \; \vec{a}_2 \ldots \vec{a}_n] \in \mathbb{R}^{n \times n}$

$a_i \in \mathbb{R}^n$

$\det(A) = V(\vec{a}_1 \; \vec{a}_2 \ldots \vec{a}_n)$

Three basic properties

I $\det(I) = V\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = 1$

II

$$\det\begin{bmatrix} a_1 & a_2 & a_3+a_3' & a_4 \end{bmatrix} = \det\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} + \det\begin{bmatrix} a_1 & a_2 & a_3' & a_4 \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3' & a_4 \end{vmatrix}$$

$$\cancel{\det} \begin{vmatrix} a_1 & a_2 & \beta a_3 & a_4 \end{vmatrix} = \beta \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix}$$

III $\begin{vmatrix} a_1 & a_2 & a_2 & a_4 \end{vmatrix} = 0$

1. Determinant of Identity Matrix

$\det(aA) = V($

$$\text{I}\quad \det(\cancel{A}\ I) = V\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = 1 .$$

2. Multilinear (Linear in each column)

$$\det\begin{bmatrix} a_1 & a_2 & a_3+a_3' & a_4 \end{bmatrix} = \det\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} + \det\begin{bmatrix} a_1 & a_2 & a_3' & a_4 \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3' & a_4 \end{vmatrix}$$

~~de~~

~~det~~ $\begin{vmatrix} a_1 & a_2 & \beta a_3 & a_4 \end{vmatrix} = \beta \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix}$

3. Identical Columns

$$\begin{vmatrix} a_1 & a_2 & a_2 & a_4 \end{vmatrix} = 0$$

Swapping Columns

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix} = -\begin{vmatrix} a_1 & a_3 & a_2 & a_4 \end{vmatrix}$$

Permutation matrix

II 21

P is a permutation matrix $\qquad \det(P) \in \{-1, 1\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

even perm. $|P| = 1$

odd perm. $|P| = -1$

one zero column

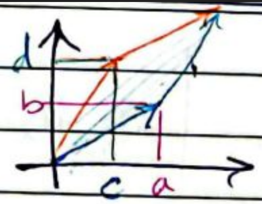
$$\det\begin{bmatrix} a_1 & a_2 & \vec{0} & a_3 \end{bmatrix} = \det\begin{bmatrix} a_1 & a_2 & -\vec{0} & a_3 \end{bmatrix}$$

$$= -\det\begin{bmatrix} a_1 & a_2 & \vec{0} & a_3 \end{bmatrix}$$

$$\Rightarrow \det\begin{bmatrix} a_1 & a_2 & \vec{0} & a_3 \end{bmatrix} = 0$$

Determinant of a 2x2 matrix

$$\det\begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}$$

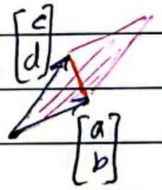


$$= \det\begin{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} & \begin{matrix} c \\ d \end{matrix} \end{bmatrix} = \begin{vmatrix} a & c \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & c \\ b & d \end{vmatrix}$$

$$= \begin{vmatrix} a & c \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & c \\ b & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ b & d \end{vmatrix}$$

$$ac\,\underbrace{\overset{a_1 = a_2}{\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}}}_{0} + ad\,\underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{1} + bc\,\underbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{-1} + bd\,\underbrace{\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}}_{0}$$

$$\det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$



Diagonal Matrix

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Diagonal Matrix

$$\begin{vmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{vmatrix} = d_1 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{vmatrix}$$

$$= d_1 d_2 d_3 d_4 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = d_1 d_2 d_3 d_4$$

Scaling a matrix

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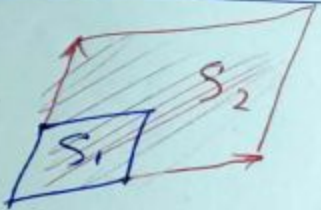
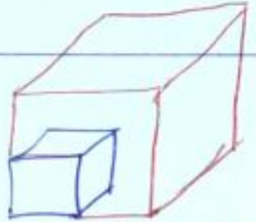
$A \in \mathbb{R}^{n \times n} \qquad \alpha \in \mathbb{R} \qquad A = [a_1 \; a_2 \cdots a_n]$

$$\det(\alpha A) = \det\left(\begin{bmatrix} \alpha a_1 & \alpha a_2 & \cdots & \alpha a_n \end{bmatrix}\right)$$

$$= \alpha^n \det(A)$$



$S_2 = 4 S_1$



$V_2 = 8 V_1$

Singular matrices

$$\left| a_1 \; a_2 \; \alpha a_1 + \beta a_2 \right| = \alpha \underbrace{\left| a_1 \; a_2 \; a_1 \right|}_{0} + \beta \underbrace{\left| a_1 \; a_2 \; a_2 \right|}_{0} = 0$$

Singular matrices

One column is a linear combination of others (some of) IV

$$\begin{vmatrix} a_1 & a_2 \cdots & a_{n-1} & a_n \end{vmatrix}$$

$$a_n = \sum_{i=1}^{n-1} \beta_i a_i$$

$$\begin{vmatrix} a_1 & a_2 \cdots & a_{n-1} & \sum_{i=1}^{n-1} \beta_i a_i \end{vmatrix} \longleftrightarrow \sum_{i=1}^{n-1} \beta_i \begin{vmatrix} a_1 & a_2 \cdots & a_{n-1} & a_i \end{vmatrix} = 0$$

Singular Matrices

Determinant of a singular matrix is zero.

(is the converse true?)

Triangular Matrices

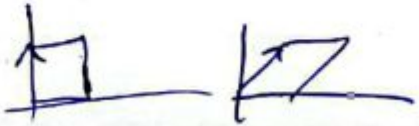
Triangular Matrices

$$
\begin{array}{c}
\begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 15 & 6 & 3 \end{vmatrix} \\
\begin{array}{ccc} a_1 & a_2 & a_3 \end{array}
\end{array}
=
\begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 15 & 0 & 3 \end{vmatrix}
=
\begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ \cancel{15}\,0 & 0 & 3 \end{vmatrix}
=
\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}
= 1 \times 2 \times 3
$$

$a_2 \mathrel{-}= 2a_3$

$a_1 \mathrel{-}= 5a_3$

$a_1 \mathrel{-}= 2a_2$



Triangular Matrices

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Triangular Matrices

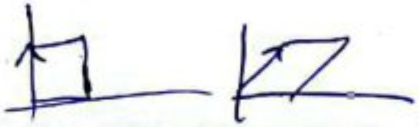
$$\begin{array}{c} \begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 15 & 6 & 3 \end{vmatrix} \\ \;\; a_1 \quad a_2 \quad a_3 \end{array} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 15 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ \cancel{15}\,0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1 \times 2 \times 3$$

$a_2 - = 2a_3$

$a_1 - = 5a_3$

$a_1 - = 2a_2$

What if a diagonal element is zero.



Triangular Matrices

A is (Upper or Lower) Triangular Matrix

$$|A| = \text{product of } \cancel{\text{dig}}\text{diagonal elements}$$

$$= \prod_{i=1}^{n} a_{ii}$$

Product of matrices

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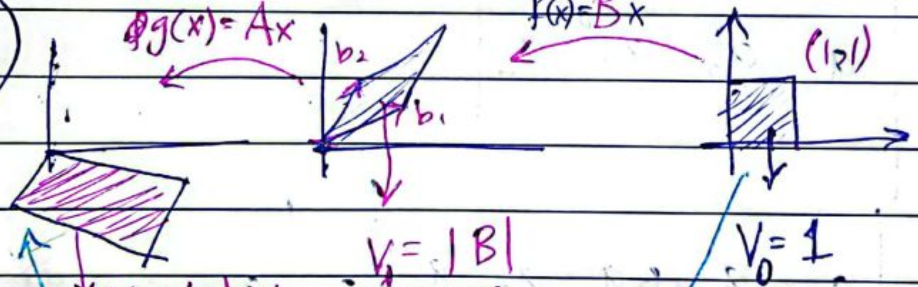
$\det(AB)$



$V_2 = |A|\, V_1 = |A|\,|B|$

$h(x) = (g \circ f)(x) = A(Bx)$
$= (AB)\,x$

$\underline{V_2 = |A\,B|}$

$\Rightarrow |A\,B| = |A|\;|B|$

Product of matrices

$\det(A\,B) = \det(A)\,\det(B)$

try proving using the previous properties.

Inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$AA^{-1} = I$$

$$\Rightarrow |AA^{-1}| = |I|$$

$$\Rightarrow |A||A^{-1}| = 1$$

Determinant of a non-singular matrix

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$\det(A)\ \det(A^{-1}) = 1 \Rightarrow$ determinant of a non-singular matrix is non-zero.

Determinant of a non-singular matrix

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$\det(A)\ \det(A^{-1}) = 1 \Rightarrow$ determinant of a non-singular matrix is non-zero.

$\Rightarrow$ $\det(A) = 0$ if and only if A is singular.

Transpose of a matrix

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$$\det(A^T) = \det(A)$$

All the determinant properties about the columns of a matrix applies to the rows of a matrix.

Eigenvalues and Eigenvectors

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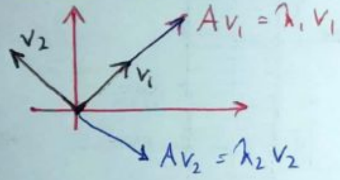
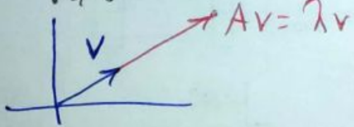


~~For A~~

For a matrix $A \in \mathbb{R}^{n\times n}$ is there a vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$

$v \neq 0$



if there exists such a vector $v \in \mathbb{R}^n$

بردار ویژه

v is called an eigenvector of A.

λ is called an eigenvalue of A.

مقدار ویژه

Eigenvalues and Eigenvectors

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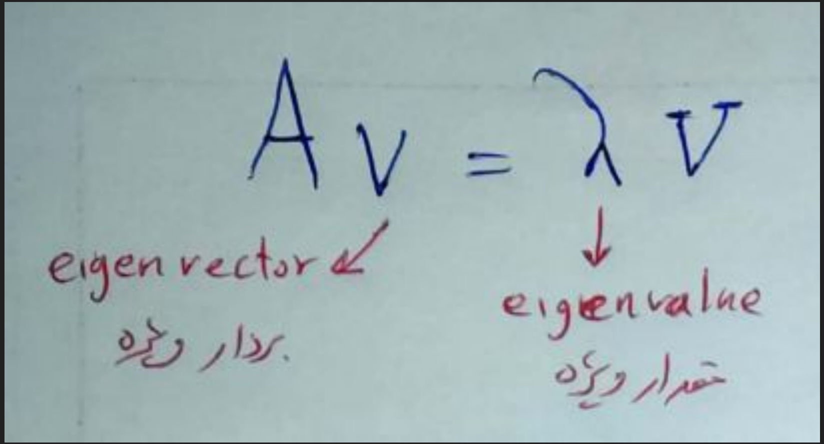
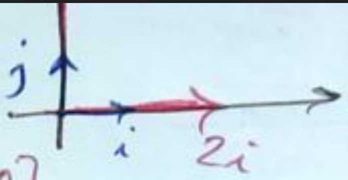


Example: Diagonal Matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$



eigenvectors are $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

eigenvalue

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 2 \right), \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 3 \right)$$

eigenvector

correspondig eigenvalue

Eigenvalues are homogeneous

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$$Av = \lambda v \Rightarrow A(2v) = \lambda(2v)$$
$$A(\alpha v) = \lambda(\alpha v)$$

for eigenvectors the orientation matters (not lenght)

Example

$$Av = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + \alpha v_2 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1-\lambda) v_1 + \alpha v_2 \\ (1-\lambda) v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} v_2 = 0 \Rightarrow (1-\lambda) v_1 = 0 \overset{v_1 \neq 0}{\Rightarrow} \lambda = 1 \\ (1-\lambda) = 0 \,,\; v_2 \neq 0 \end{cases}$$

$$\Downarrow$$

$$\alpha v_2 = 0$$

$$\Downarrow \text{ only possible if } \alpha = 0$$

$$\Rightarrow A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \text{ with } \alpha \neq 0 \text{ only has a single eigenvector } \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Example: Identity matrix

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$A = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Choose any $v \neq 0 \in \mathbb{R}^n$

$A = I \in \mathbb{R}^{n \times n}$

$$Av = Iv = v \Rightarrow Av = 1\,v$$

eigen value

Any vector $v \neq \emptyset$ is an eigen vector of I.

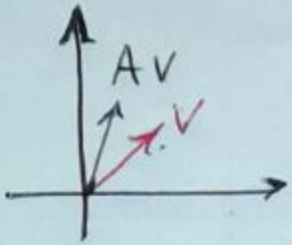
was in any case 1 is an eigenvalue.

Example: 2D rotation

eig 1

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

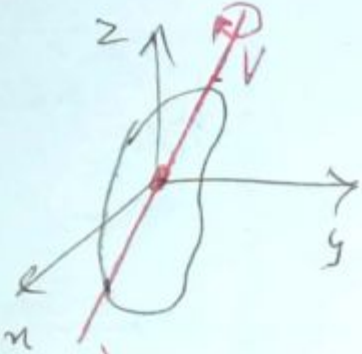
$\theta \neq 0$



No $\underbrace{}_{\text{(real)}}$ Eigenvectors

Example: 3D Rotation

$R \in \mathbb{R}^{3\times 3}$ is $\underbrace{\text{a}}_{\text{3D}}$ rotation matrix



$Rv = 1 \cdot v$

$(v, 1)$ only (real) eigenpair

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axis of rotation

Singular matrices

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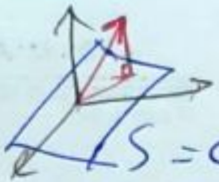
Let $A \in \mathbb{R}^{n \times n}$ be singular. $\Rightarrow \text{Dim}(N(A)) > 0$

$\exists\, v \in N(A) \quad v \neq 0 \qquad A\vec{v} = \vec{0} = 0 \cdot \vec{v}$

Any $v \neq 0$ in $N(A)$ is an eigenvector of A with the corresponding eigenvalue $\lambda = 0$.

Projection matrix

Projection matrix P



$S = C(P)$

$P = V \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} V^{-1}$

$v \in S \Rightarrow Pv = v \qquad \lambda = 1$

$v \in S^{\perp} \Rightarrow Pv = 0 = 0 . v \Rightarrow \lambda = 0$

1. $P^2 = PP = P \qquad Pv = \lambda v \Rightarrow \left. \begin{aligned} P^2 v &= PPv = Pv = \lambda v \\ &= P(\lambda v) = \lambda^2 v \end{aligned} \right\} \begin{aligned} &\lambda = \lambda^2 \Rightarrow (\lambda^2 - \lambda) = 0 \\ &\lambda = 0, \ \lambda = 1 \end{aligned}$

Computing Eigenvalues

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Eig 2 (I)

$A \in \mathbb{R}^{n\times n}$

$A\underline{v} = \underline{\lambda}\,\underline{v} \qquad v \in \mathbb{R}^n,\ \lambda \in \mathbb{R}$

$Av = \lambda v \Rightarrow Av - \lambda v = 0 \Rightarrow \underset{n\times n}{A}v - \underset{n\times n}{(\lambda I)}v = 0$

$\Rightarrow \overbrace{(A - \lambda I)}^{n\times n} v = 0 \Longrightarrow (A - \lambda I)$ is singular

$v \neq 0$

$\det(A - \lambda I) = 0 \Longrightarrow$ a polynomial on λ of degree n

چندجمله‌ای درجه n

Computing Eigenvalues

$$Av = \lambda v \Rightarrow Av = (\lambda I)v \Rightarrow Av - \lambda I v = 0$$

$$(A - \lambda I)v = 0 \Rightarrow (A - \lambda I) \text{ has a non-zero null vector} \Rightarrow |A - \lambda I| = 0$$

$$\left(A - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) v = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} v = 0$$

Example

$$A=\begin{bmatrix}0&3\\3&0\end{bmatrix} \Rightarrow A-\lambda I=\begin{bmatrix}0&3\\3&0\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}=\begin{bmatrix}\lambda&3\\3&\lambda\end{bmatrix}$$

charcteristic c
چندجمله‌ای مشخصه

$$\det\begin{bmatrix}\lambda&3\\3&\lambda\end{bmatrix}=0 \Rightarrow \underline{\lambda^2-3^2}=0 \Rightarrow (\lambda-3)(\lambda+3)=0$$

چندجمله‌ای درجه ۲

$$\lambda=3 \Rightarrow \begin{bmatrix}0&3\\3&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=3\begin{bmatrix}v_1\\v_2\end{bmatrix} \Rightarrow \begin{bmatrix}3v_2\\3v_1\end{bmatrix}=\begin{bmatrix}3v_1\\3v_2\end{bmatrix} \Rightarrow v_1=v_2$$

$$v=\alpha\begin{bmatrix}1\\1\end{bmatrix}$$

$$\lambda=-3 \Rightarrow \begin{bmatrix}0&3\\3&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=-3\begin{bmatrix}v_1\\v_2\end{bmatrix} \Rightarrow \begin{bmatrix}3v_2\\3v_1\end{bmatrix}=\begin{bmatrix}-3v_1\\-3v_2\end{bmatrix} \Rightarrow v_1=-v_2$$

$$v=\alpha\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\left(\begin{bmatrix}1\\1\end{bmatrix},3\right),\ \left(\begin{bmatrix}-1\\1\end{bmatrix},-3\right)$$

choose eigenvector to be unit vectors

$$\left(\begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\end{bmatrix},3\right),\ \left(\begin{bmatrix}-\sqrt{2}/2\\\sqrt{2}/2\end{bmatrix},3\right)$$

Example

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eig2 II

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix}$$

$$\underbrace{(2-\lambda)^2 - 3^2}_{\text{charcteristic polynomial of } A} = 0 \Rightarrow 2-\lambda = \pm 3 \Rightarrow \begin{cases} \lambda = -1 \\ \lambda = 5 \end{cases}$$

$$\lambda = -1 \Rightarrow A - \lambda I \Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 5 \Rightarrow A - \lambda I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$