Homework 1

Write the following in LaTeX.

Proposition 1 Consider the linear map $f: \mathbb{R}^n \to \mathbb{R}^n$ defined as $f(\vec{x}) = A\vec{x}$ with $A \in \mathbb{R}^{n\times n}$. If A is non-singular then f is a bijection.

To prove this we use the following lemma.

Lemma 1. If $A \in \mathbb{R}^{n\times n}$ is non-singular then $A\vec{x} = \vec{0}$ implies $\vec{x} = \vec{0}$ for $\vec{x} \in \mathbb{R}^n$.

Proof Let $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$ where $\vec{a}_i \in \mathbb{R}^n$ is the i-th column of A. Then

$$A\vec{x} = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i \vec{a}_i. \qquad (1)$$

As A is nonsingular, $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are linearly independent. Thus, $\sum_{i=1}^{n} x_i \vec{a}_i = 0$ implies $x_1 = x_2 = \cdots = x_n = 0$, giving $\vec{x} = \vec{0}$. $\square$

Proof of Proposition 1 First we prove that f is one-to-one. Assume that $f(\vec{x}_1) = f(\vec{x}_2)$ for $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$. This gives $A\vec{x}_1 = A\vec{x}_2$, and thus $A(\vec{x}_1 - \vec{x}_2) = \vec{0}$. Now from Lemma 1 we have $\vec{x}_1 - \vec{x}_2 = 0$ and hence $\vec{x}_1 = \vec{x}_2$, proving that f is one-to-one.

Now, $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are n linearly independent vectors in $\mathbb{R}^n$ and thus span $\mathbb{R}^n$. This means that any vector $\vec{y} \in \mathbb{R}^n$ can be written as $\vec{y} = \sum_{i=1}^{n} x_i \vec{a}_i$ for some scalars $x_1, x_2, \ldots, x_n \in \mathbb{R}$. As we obsered in (1) this means that any vector $\vec{y} \in \mathbb{R}^n$ can be written as

$$\vec{y} = A\vec{x} = f(\vec{x}), \qquad (2)$$

for some $\vec{x} \in \mathbb{R}^n$. This ~~p~~ proves that f is onto. The functio f is one-to-one and onto, and therefore a bijection. $\square$

You have to follow the following rules:

- Your document must contain **a title, a date, and your name as the author**. Also, write your affiliation as "K. N. Toosi University of Technology".
- You **MUST** replicate the original text above (you may correct typos). Do not write other proofs of the proposition.
- Write all scalars with regular italic letters (a, b, α, β.)
- Write the fundamental sets (ℝ,ℤ,...) using blackboard bold letter (\mathbb{R}).
- Represent *vectors* with bold lower-case letters (e.g. **a**, using \mathbf{a}). **Do not put arrows above the vectors like I did in the original text above.**
- Represent matrices either with bold upper-case letters (e.g. **A**, using \mathbf{A}), or with typewriter upper-case letters (e.g. `A`, using \mathtt{A}).
- You can define macros to make your life easier (for instance **\newcommand{\Real}{\mathbb{R}}**)
- Use the equation, align, or similar environments for equations (1) and (2).
- You **MUST** use the theorem environments in LaTeX to write propositions, lemmas, and proofs. For more information, visit [this link](), and research online to understand the differences between a theorem (قضیه), a proposition, and a lemma.
- You need to use proper cross-referencing in your latex file.
    - Wrong: `From Lemma 1.`
    - Right: `From Lemma \ref{?}`
    - Wrong: `As we observe in (1).`
    - Right: `As we observe in \eqref{?}`
- Ensure that the mathematical entities like functions, vectors, scalars, etc., are put inside a math environment:
    - Wrong: `The function f is one-to-one and onto.`
    - Right: `The function $f$ is one-to-one and onto.`
    - Wrong: `The i-th column of A.`
    - Right: `The $i$-th column of $\mathtt{A}$.`

- Submit two files named **homework1.pdf** file and a **homework1.tex** file.
- You will also present the homework to the TAs. **You have to be able to explain the latex commands you used to the TAs. You must be able to make modifications to your document as requested by the TA.** Therefore, ensure that your submission is entirely your own work.