Mathematics for AI
Homework 2

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Read these first:

i To achieve the full score, you need to write your solutions using LaTeX. If you choose to write your solutions on paper or in a word processing software (e.g., MS Word, LibreOffice), you can receive up to 90% of the score.

ii If writing on paper, you must use a scanner or a camera scanning app (e.g., CamScanner) to scan the document and submit it as a *single* PDF file. Ensure your answers are written neatly, organized, and legible on paper.

iii When using LaTeX, follow one of these two conventions:

(a) Represent scalars with italic letters (a, A), vectors with bold lowercase letters ($\mathbf{a}$, using `\mathbf{a}`), and matrices with bold uppercase letters ($\mathbf{A}$, using `\mathbf{A}`), or

(b) Represent scalars with italic letters (a, A), vectors with bold letters ($\mathbf{a}$, $\mathbf{A}$), and matrices with typewriter uppercase letters ($\mathtt{A}$, using `\mathtt{A}`).

iv Your LaTeXdocument must include a *title*, a *date*, and your name as the *author*.

v If writing on paper, submit a *single* PDF file; do not send multiple image files.

vi If using LaTeX, submit the *.tex* source file (along with any other required source files) in addition to the PDF file.

Here is a short tutorial on LaTeX: `https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`

Questions For each questions, you may use the results of the previous questions (but not the following questions).

Matrices

1. Consider a matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$, such that $\mathtt{A}\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}$. Prove that $\mathtt{A} = \mathtt{0}_{m\times n}$, that is all the entries of $\mathtt{A}$ are zero.

2. Consider a matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$, such that $\mathtt{A}\,\mathbf{x}_i = 0$ for $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n \in \mathbb{R}^n$, where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ form a basis for $\mathbb{R}^n$. Prove that $\mathtt{A} = 0_{m\times n}$.

3. Consider a square matrix $\mathtt{A} \in \mathbb{R}^{n\times n}$ for which $\mathtt{A}\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that $\mathtt{A} = \mathtt{I}_n$, the n by n identity matrix.

4. Give an example of a matrix $\mathtt{A} \in \mathbb{R}$, such that $\mathtt{A}\mathbf{x} = \mathbf{x}$ for some nonzero vector $\mathbf{x} \in \mathbb{R}^n$, $\mathtt{A}$ is not the identity matrix.

5. Assume that the vectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$ are linearly independent. Prove that the set of vectors $\mathbf{a}'_1, \mathbf{a}_2, ..., \mathbf{a}_n$ are also linearly independent where $\mathbf{a}'_1 = \mathbf{a}_1 + \beta\mathbf{a}_2$ for some scalar β.

6. Consider two matrices $\mathtt{A} \in \mathbb{R}^{m\times n}$ and $\mathtt{B} \in \mathbb{R}^{n\times p}$. Prove that $\mathcal{C}(\mathtt{A}) \subseteq \mathcal{C}(\mathtt{AB})$, where $\mathcal{C}(\cdot)$ represents the column space.

7. Consider a matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$ and a square *invertible* matrix $\mathtt{B} \in \mathbb{R}^{n\times n}$. Prove that $\mathcal{C}(\mathtt{A}) = \mathcal{C}(\mathtt{AB})$. (Hint: to prove that two sets S_1 and S_2 are equation you can show $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$).

8. Consider two matrices $\mathtt{A} \in \mathbb{R}^{m\times n}$ and $\mathtt{B} \in \mathbb{R}^{n\times p}$ where $\mathtt{B}$ has *full row rank* (i.e. $\text{rank}(\mathtt{B}) = n$). Prove that $\mathcal{C}(\mathtt{A}) = \mathcal{C}(\mathtt{A\,B})$.

Matrix Multiplication

9. Consider the matrices $\mathtt{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] \in \mathbb{R}^{m\times n}$, $\mathtt{D} = \text{diag}([d_1, d_2, \ldots, d_n]) \in \mathbb{R}^{n\times n}$, $\mathtt{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n] \in \mathbb{R}^{p\times n}$, where $\mathtt{D}$ is a diagonal matrix with diagonal elements d_i. Show that

$$\mathtt{ADB}^T = \sum_{i=1}^{n} d_i\, \mathbf{a}_i \mathbf{b}_i^T$$

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Linear Equations

To answer the following questions you need to use the fact that the set of solutions to a system of linear equations $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ is in the form of $\{\mathbf{x}_p + \mathbf{x}_n \mid \mathbf{x}_n \in \mathcal{N}(\mathtt{A})\}$, where $\mathbf{x}_p$ is a particular solution.

10. Let $\mathtt{A} \in \mathbb{R}^{m\times n}$ be a fat matrix (i.e. $m < n$) with *full row rank* and $\mathbf{b} \in \mathbb{R}^m$. Show that $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

11. Let $\mathtt{A} \in \mathbb{R}^{m\times n}$ be a tall matrix (i.e. $m > n$) with *full column rank* and $\mathbf{b} \in \mathbb{R}^m$. Show that $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ has either no solution or exactly one solution.

12. Consider the system of linear equations $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ with $\mathtt{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$, and let $\mathcal{S}$ be the set of solutions to it. Show that

   (a) $\mathcal{S}$ is a linear subspace if and only if $\mathbf{b} = \mathbf{0}$.

   (b) If $\mathcal{S}$ is nonempty, then there exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that the set $\{\mathbf{z} - \mathbf{y} \mid \mathbf{z} \in \mathcal{S}\}$ is a linear subspace.

Projections

13. Consider a linear subspace $\mathcal{S}$ and a vector $\mathbf{y} \in \mathcal{S}$. Using the projection formula, show that the projection of $\mathbf{y}$ into $\mathcal{S}$ is itself.

14. For a linear subspace $\mathcal{S} \subseteq \mathbb{R}^n$ its *orthogonal complement* is defined as $\mathcal{S}^\perp = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y}^T\mathbf{x} = 0 \text{ for all } \mathbf{x} \in \mathcal{S}\}$. In other words, $\mathcal{S}^\perp$ comprises all the vectors that are perpendicular to all vectors in $\mathcal{S}$. Show that the orthogonal complement of a linear subspace is a linear subspace.

15. Prove that the *null space* of a matrix is the orthogonal complement of its *row space*.

Determinant

16. Show that $\det(\mathtt{A}^{-1}) = 1/\det(\mathtt{A})$.

17. Prove that the determinant of an orthogonal matrix is either equal to 1 or -1. (Hint: use the definition of an orthogonal matrix.)

18. Show that the determinant of a projection matrix is either equal to 0 or 1. (Hint: remember that projections are *idempotent*.) How do you explain this geometrically?

Eigenvalues and Eigenvectors

19. What is the relation between the eigenvalues and eigenvectors of the square matrix $\mathtt{A}$ and those of $\mathtt{A} - \alpha\mathtt{I}$ where $\alpha \in \mathbb{R}$ and $\mathtt{I}$ is the identity matrix?

20. Prove that any eigenvalue of $\mathtt{A}$ is also an eigenvalue of $\mathtt{A}^T$. (Hint: use the characteristic polynomial).

21. The square matrix $\mathtt{A}$ is called (left) stochastic (or a Markov matrix) if its elements are nonnegative and its columns add up to 1 (programmatically `sum(A,axis=0) == ones((1,n))`). Prove that $\mathtt{A}$ has at least one unit eigenvalue $\lambda = 1$. (Hint: First prove that $\mathtt{A}^T$ has a unit eigenvalue.)

22. Let $\mathbf{v}$ be an eigenvector of $\mathtt{A}$ with a nonzero corresponding eigenvalue $\lambda \neq 0$. Prove that $\mathbf{v}$ is in the column space of $\mathtt{A}$.

23. Let $\mathtt{A}$ be a real symmetric matrix with real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$. Prove that if $\lambda_i \neq \lambda_j$ then $\mathbf{v}_i \perp \mathbf{v}_j$.