$$C = AB$$

$\mathrm{rank}(C)$ $\mathrm{rank}(A)$ $\mathrm{rank}(B)$

$$\underset{m\times p}{C} = \underset{m\times n}{A}\ \underset{n\times p}{B} \Rightarrow [c_1\ c_2 \cdots c_p] = A[b_1\ b_2 \cdots b_p]$$

$$\mathcal{C}(AB) \subseteq \mathcal{C}(A)$$

$$x \in \mathcal{C}(AB) \Rightarrow \exists\, y \in \mathbb{R}^p \quad x = (AB)y = A(\underbrace{By}_{z \in \mathbb{R}^n}) = Az$$

$$\Rightarrow x \in \mathcal{C}(A) \quad \Rightarrow \mathcal{C}(AB) \subseteq \mathcal{C}(A)$$

$$\dim(\mathcal{C}(AB)) \leq \dim(\mathcal{C}(A))$$

$$\mathrm{rank}(AB) \leq \mathrm{rank}(A)$$

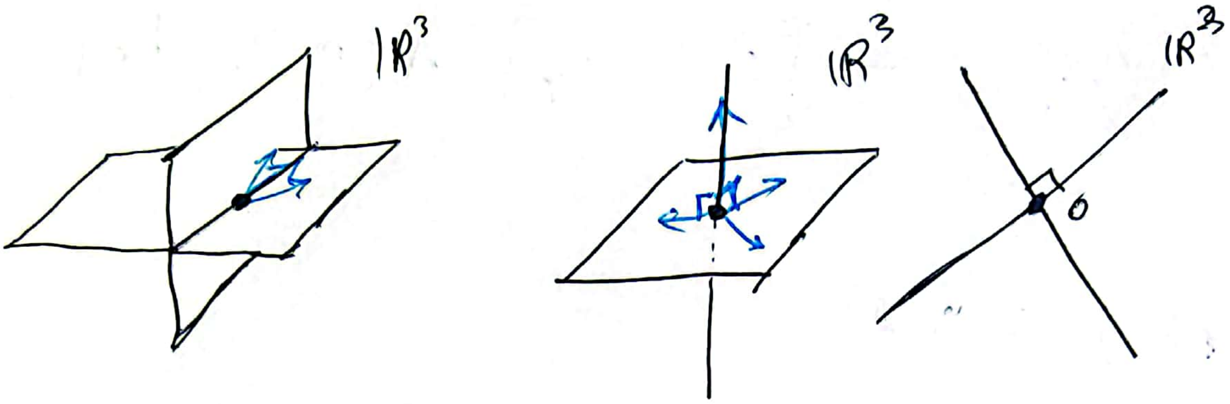
same argument $\Rightarrow \mathrm{rank}(AB) \leq \mathrm{rank}(B)$

$$\prod_{i=1}^{N} A_i = A_1 A_2 \cdots A_N$$

$$\mathrm{rank}\left(\prod_{i=1}^{N} A_i\right) \leq \min_i \ \mathrm{rank}(A_i)$$

S_1, S_2 linear subspaces

$S_1 \perp S_2 \quad \forall\, x_1 \in S_1,\, x_2 \in S_2 \Rightarrow x_1 \perp x_2$



$A \in \mathbb{R}^{m \times n} \qquad C(A) \subseteq \mathbb{R}^m$

$C(A^T) = R(A) \subseteq \mathbb{R}^n$

$\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0\} = N(A) \subseteq \mathbb{R}^n$

$N(A) \overset{?}{\perp} R(A) \qquad x \in N(A) \Rightarrow Ax = \vec{0}$

$y \in R(A) = C(A^T) \Rightarrow y = A^T z$

$\langle x, y \rangle = y^T x = (A^T z)^T x = z^T A x = z^T \vec{0} = 0$

$x \in N(A)$

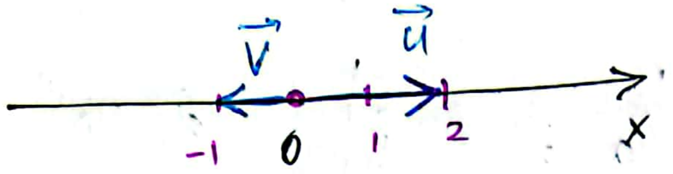
$\Rightarrow \quad N(A) \perp R(A)$

$N(A^T) = \{x \mid \vec{x}^T A = \vec{0}^T\} = \{\vec{x} \mid A^T \vec{x} = \vec{0}\}$ left null space

$N(A^T) \perp C(A)$

Signed Volume

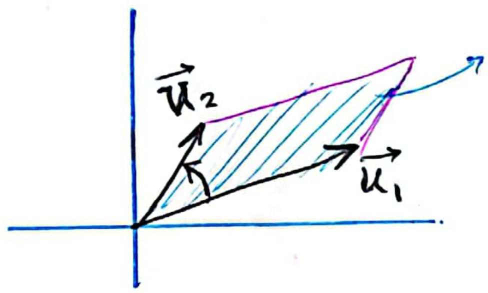
1D: signed lenght



$V(\vec{u}) = 2$

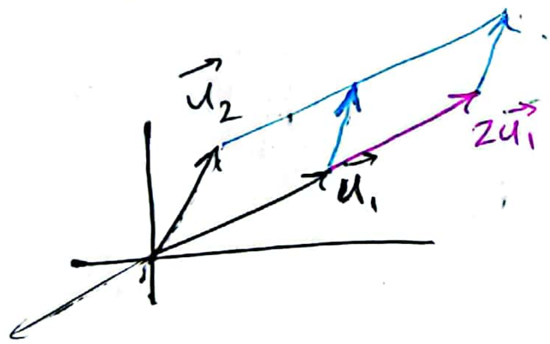
$V(\vec{v}) = -1$

2D:



$V(\vec{u}_1, \vec{u}_2) > 0$

$V(\vec{u}_2, \vec{u}_1) = -V(\vec{u}_1, \vec{u}_2)$

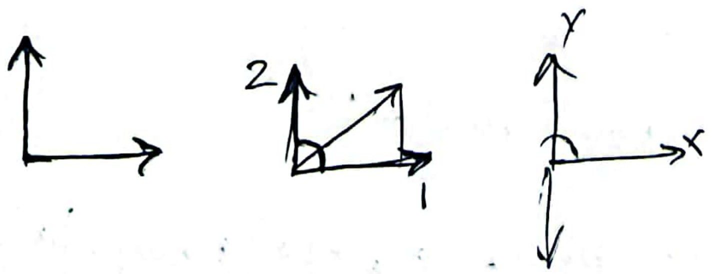


$V(2\vec{u}_1, \vec{u}_2) = 2V(\vec{u}_1, \vec{u}_2)$

$V(\vec{u}+\vec{v}, \vec{w}) = V(\vec{u}, \vec{w}) + V(\vec{v}, \vec{w})$

$V(\vec{u}_1 - \vec{u}_1, \vec{u}_2) = V(u_1, u_2) - V(u_1, u_2)$

$V(\cdot, \cdot)$ bilinear



$\det(A) = |A|$

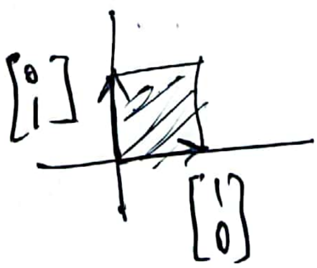
$A \in \mathbb{R}^{n \times n}$ $\quad A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & - & a_n \\ | & | & & | \end{bmatrix}$

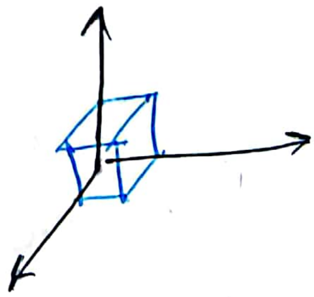
$\det(A) = |A| = |\vec{a}_1\ \vec{a}_2 - \vec{a}_n|$ $\quad \vec{a}_i \in \mathbb{R}^n$

$$\left| a_1, \overset{a_2}{\left(\sum \alpha_i \vec{b}_i\right)}, a_3 \cdots, a_n \right| = \sum \alpha_i \left| \vec{a}_1\ \vec{b}_i\ a_3 - a_n \right|$$

$\det : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$

$\det(I)$





$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$

$\det(I) = 1$

$$|a_1\ a_1\ a_3 \cdots a_n| = 0$$

$\cancel{|a\ b\ b|} = |a\ \ b+c\ \ b+c| = 0$

$= |a\ \ b\ \ b+c| + |a\ \ c\ \ b+c| = |a\ b\ c| + \overset{0}{\cancel{|a\ b\ b|}} + |a\ c\ b| + \overset{0}{\cancel{|a\ c\ c|}}$

$A \in \mathbb{R}^{n \times n}$ $\quad \det(\alpha A) = \alpha^n \det(A)$ $\quad$ MA10 (V)

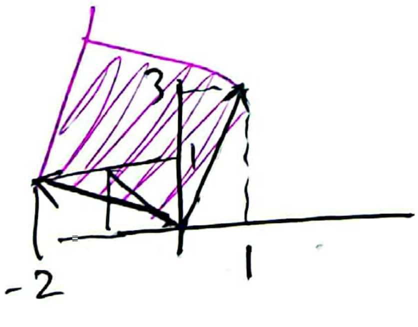
$$\begin{vmatrix} u & v & w \end{vmatrix} = \begin{vmatrix} u & v & \alpha u + \beta v \end{vmatrix}$$

$u, v, w \in \mathbb{R}^3$

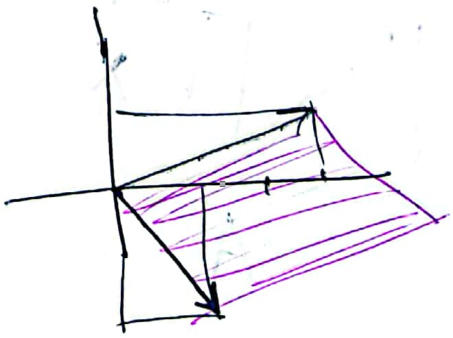
$$= \alpha \cancel{\begin{vmatrix} u & v & u \end{vmatrix}}^{\,0} + \beta \cancel{\begin{vmatrix} u & v & v \end{vmatrix}}^{\,0} = 0$$

$A \in \mathbb{R}^{n \times n}$ singular $\Longrightarrow \det(A) = 0$

$$\begin{vmatrix} \vec{u} + a\vec{v} & \vec{v} & \vec{w} \end{vmatrix} = \begin{vmatrix} \vec{u} & \vec{v} & \vec{w} \end{vmatrix} + a \cancel{\begin{vmatrix} \vec{v} & \vec{v} & w \end{vmatrix}}^{\,0}$$

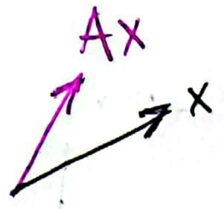


$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

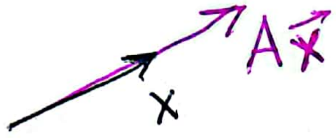


$A \in \mathbb{R}^{n\times n}$ $\quad$ $f: \mathbb{R}^n \to \mathbb{R}^n$

$$f(\vec{x}) = A\vec{x}$$



$x \qquad Ax$



$$\boxed{Ax} \qquad A \in \mathbb{R}^{n\times n} \quad x \in \mathbb{R}^n$$

$$O(n^2)$$

$$A = \begin{bmatrix} \alpha_1 & & 0 \\ & \alpha_2 & \\ 0 & & \ddots \ \alpha_n \end{bmatrix} \qquad Ax = \begin{bmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \ddots \ \alpha_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 x_1 \\ \alpha_2 x_2 \\ \vdots \\ \alpha_n x_n \end{bmatrix}$$

$$O(n)$$

$$\underbrace{V \begin{bmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \ddots \ \alpha_n \end{bmatrix} V^{-1}}_{A} x$$

$$Av_1 = \lambda_1 v_1$$

$$Av_2 = \lambda_2 v_2$$

$A \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^n$ $\lambda \in \mathbb{R}$ MA 10 VII

$$AV = \lambda V$$

یک بردار ویژه A

Eigenvetor

مقدار ویژه متناظر V

Eigenvalue

$$A(\alpha V) = \alpha A V = \lambda (\alpha V)$$

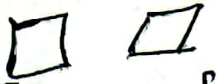
$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Eigenvector

corresponding eigenvalue

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 2 \right)$$ Eigenpair

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 3 \right)$$



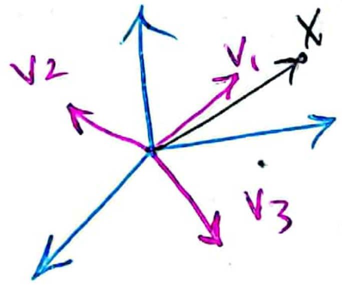
$$\begin{bmatrix}1&3\\0&1\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=\lambda\begin{bmatrix}x\\y\end{bmatrix}$$

$$\begin{bmatrix}x+3y\\y\end{bmatrix}=\begin{bmatrix}\lambda x\\\lambda y\end{bmatrix}\qquad y=\lambda y\begin{cases}\lambda=1\\y=0\end{cases}$$

$$\lambda=1\Rightarrow\begin{bmatrix}x+3y\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}\Rightarrow 3y=0\Rightarrow y=0$$

$$y=0\Rightarrow\begin{bmatrix}x\\0\end{bmatrix}=\lambda\begin{bmatrix}x\\0\end{bmatrix}\Rightarrow\left(\begin{bmatrix}1\\0\end{bmatrix},1\right)$$

A



$$\vec{x}=y_1\vec{v_1}+y_2\vec{v_2}+y_3\vec{v_3}\qquad\vec{Y}=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$$

$$=\underbrace{\begin{bmatrix}\vec{v_1}&\vec{v_2}&\vec{v_3}\end{bmatrix}}_{V}\underbrace{\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}}_{Y}=V\vec{Y}\qquad\vec{Y}=V^{-1}\vec{x}$$

$$A\vec{x}\equiv D\vec{Y}=\begin{bmatrix}\lambda_1&&\\&\lambda_2&\\&&\lambda_3\end{bmatrix}\vec{Y}$$

$$A\vec{x}=\underbrace{V\,D\,V^{-1}}_{A}\vec{x}\qquad A\vec{v_1}\qquad \overset{\lambda_1 v_1}{\Uparrow}$$

$$A\vec{v_1}=VDV^{-1}v_1=VD\begin{bmatrix}1\\0\\0\end{bmatrix}=V\begin{bmatrix}\lambda_1&&\\&\lambda_2&\\&&\lambda_3\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}v_1&v_2&v_3\end{bmatrix}\begin{bmatrix}\lambda_1\\0\\0\end{bmatrix}$$

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$\vec{x} \in \mathbb{R}^n \qquad A \in \mathbb{R}^{n \times n}$ MA IV III

$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ n-independent eigenvetors of A

$$x = \sum \alpha_i \vec{v}_i = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n$$

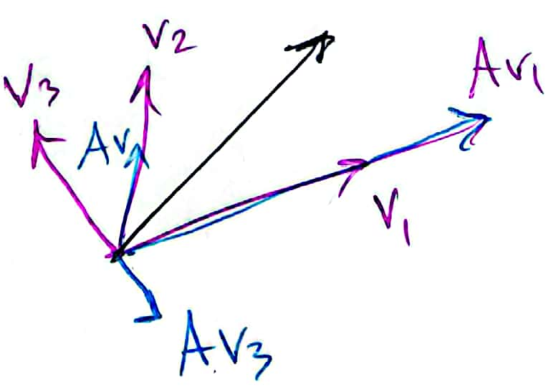
$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ | \\ \alpha_n \end{bmatrix}$$

$$Ax = \alpha_1 A\vec{v}_1 + \alpha_2 A\vec{v}_2 + \dots + \alpha_n A\vec{v}_n$$

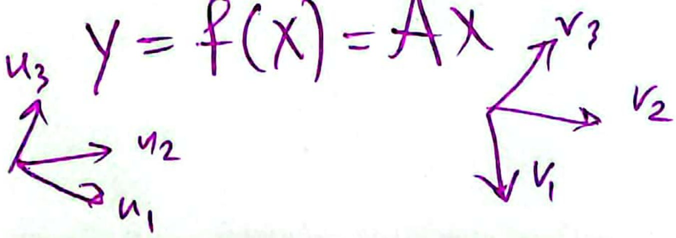
$$= \alpha_1 \lambda_1 \vec{v}_1 + \alpha_2 \lambda_2 \vec{v}_2 + \dots + \alpha_n \lambda_n \vec{v}_n$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 \alpha_1 \\ \lambda_2 \alpha_2 \\ | \\ \lambda_n \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & v_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \lambda_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ | \\ \alpha_n \end{bmatrix}}$$



$y = f(x) = Ax$



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