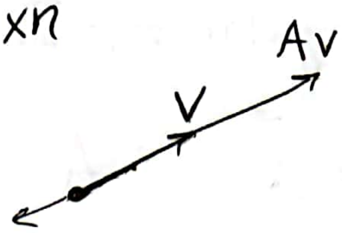
$A\vec{v} = \lambda \vec{v}$ $\quad A \in \mathbb{R}^{n\times n}$

$\lambda \rightarrow$ مقدار ویژه $\quad \vec{v} \rightarrow$ بردار ویژه.



$\alpha v \equiv \vec{v}$ $\quad \vec{v} \neq 0$ $\quad \|v\| = 1$

$\pm v$

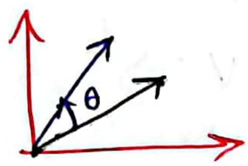
$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I\vec{v} = 1 \cdot \vec{v}$

Every ~~vect~~ nonzero vector in $\mathbb{R}^n$ is an eigenvector of I_n.

$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\quad \theta \neq k\pi$

$R\vec{v} = \lambda\vec{v}$

$\rightarrow$ no real eigenpair



3D rotation: ~~Re~~ $R \in \mathbb{R}^{3\times 3}$

$R^T R = I \quad \det(R) = 1$

$R\vec{u} = \underline{1} \cdot \vec{u}$

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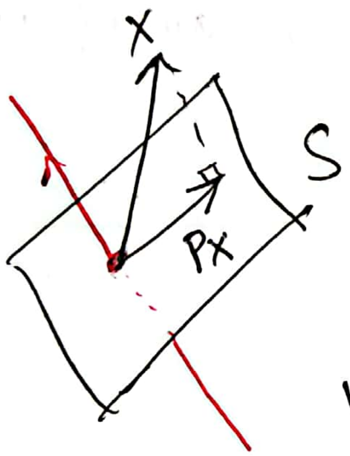
A singular $\Rightarrow \exists\, \vec{v} \neq 0,\ A\vec{v} = \vec{0}$

$N(A) \neq \{\vec{0}\}$ A has non-trivial null space

$$A\vec{v} = \vec{0} \Rightarrow A\vec{v} = 0 \cdot \vec{v}$$

Every $\vec{v} \in N(A) \setminus \{\vec{0}\}$ is a eigenvector with corresponding eigenvalue 0.



Projection matrix P

$$P^T = P$$

$$PP = P$$

$v \in S \Rightarrow Pv = 1 . v \quad (v, 1)$

$v \in S^{\perp} \Rightarrow Pv = 0 . v \quad (v, 0)$

Let v be an eigenvector of P

$$PPv = Pv$$

$$P\lambda v = \lambda v \Rightarrow \lambda^2 v = \lambda v \Rightarrow (\lambda^2 - \lambda)\vec{v} = 0$$

$$\vec{v} \neq 0 \Rightarrow \lambda(1-\lambda) = 0 \quad \begin{cases} \lambda = 0 \\ \lambda = 1 \end{cases}$$

$A\vec{v} = \lambda\vec{v} \qquad \vec{v} \in \mathbb{R}^n,\ \vec{v} \neq 0$

$$A\vec{v} - \lambda\vec{v} = 0 \Rightarrow \underbrace{A}_{n\times n}\vec{v} - \underbrace{(\lambda I)}_{n\times n} v \Rightarrow \underbrace{(A - \lambda I)}_{n\times n} \vec{v} = 0$$

$\Rightarrow (A - \lambda I)$ singular $\Rightarrow \det(A - \lambda I) = 0$

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$\det(A - \lambda I)$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (a_{11}-\lambda)(a_{22}-\lambda) - a_{21}a_{12}$$

$$= \lambda^2 + b\lambda + c \qquad \begin{cases} \lambda_1 = \checkmark \\ \lambda_2 = \checkmark \end{cases}$$

چندجمله‌ای مشخصه

charcteristic Polynomial

$$\lambda_1, \lambda_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

$v_1 \in N(A - \lambda_1 I)$

$v_2 \in N(A - \lambda_2 I)$

$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix}$

$$(2-\lambda)^2 - 9 = 4 - 4\lambda + \lambda^2 - 9 \Rightarrow$$

$$\lambda^2 - 4\lambda - 5$$

$$(\lambda-5)(\lambda+1) = 0 \begin{cases} \lambda = 5 \\ \lambda = -1 \end{cases}$$

$\lambda_1 = 5 \qquad A - \lambda_1 I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \qquad \left(5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$\lambda = -1 \qquad A - \lambda_1 I \qquad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0 \qquad \left(-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

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Eigenspace

Let λ be an eigenvalue of $A \in \mathbb{R}^{n\times n}$:

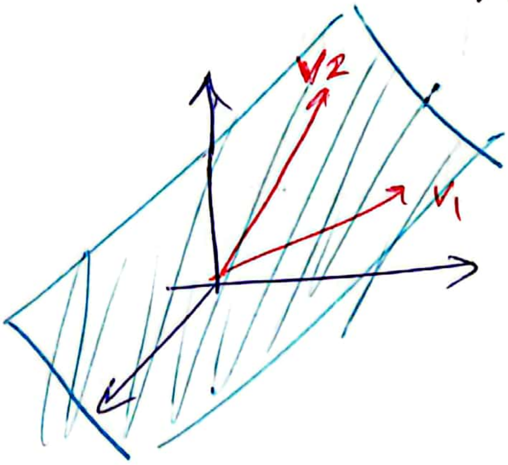
$$\mathcal{E}(\lambda) = \{v \mid Av = \lambda v\}$$

is $\mathcal{E}(\lambda)$ is a linear subspace of $\mathbb{R}^n$?

$$\vec{v} \in \mathcal{E}(\lambda) \Rightarrow Av = \lambda v \Rightarrow A(\alpha v) = \lambda(\alpha v) \Rightarrow \alpha\vec{v} \in \mathcal{E}(\lambda)$$

$$\vec{v}_1, \vec{v}_2 \in \mathcal{E}(\lambda) \Rightarrow \begin{matrix} Av_1 = \lambda v_1 \\ Av_2 = \lambda v_2 \end{matrix} \Rightarrow A(\vec{v}_1 + \vec{v}_2) = \lambda(v_1 + v_2)$$

$$\Rightarrow \vec{v}_1 + \vec{v}_2 \in \mathcal{E}(\lambda)$$



$\mathcal{E}(\lambda)$ is the corresponding eigenspace of λ.

$$\mathcal{E}(\lambda) = \{v \mid Av = \lambda v\} = \{v \mid (A - \lambda I)v = 0\}$$
$$= \mathcal{N}(A - \lambda I) \quad \text{linear}$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ~~eigen vector~~ eigenvalue $\lambda = 1$ $\quad \mathcal{E}(\lambda) = \mathbb{R}^2 \Rightarrow$

Geometric Multiplicity of $\lambda = 1 \Leftarrow \dim(\mathcal{E}(\lambda)) = 2$

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 = (1-\lambda)(1-\lambda)$$

ریشه مضاعف $\lambda = 1$ repeated root / $\lambda = 1$ repeated 2 times

algebraic multiplicity

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$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \lambda = 2$$

$$\lambda_1 = 2 \Rightarrow \text{Eigenspace} = \left\{ \alpha\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix} \right\}$$

$\dim(\mathcal{E}(2)) = 2$ geometric mult. $= 2$

algebraic mult. $= 2$

$$\lambda_2 = 3 \qquad \mathcal{E}(3) = \left\{ \alpha\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

geometric mult. $= 1$

alg mult $= 1$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (2-\lambda)^2(3-\lambda)$$

$\downarrow$ $\lambda = 2$, $\lambda = 2$ $\qquad \downarrow$ $\lambda = 3$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad \det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 3 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$$

$\lambda = 1$ algebraic mult $(\lambda = 1) = 2$.

$$\begin{bmatrix} 1-1 & 3 \\ 0 & 1-1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad N\left(\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\right) = \left\{ \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

geo-mult $= 1$

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$R(\theta) \quad \theta = \pi/2$

MA II ~~0~~ (VI)

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$ no real root!

$$\Rightarrow \lambda = \pm i \qquad \lambda_1 = +i, \quad \lambda_2 = -i$$

$$\lambda_1 = +i \Rightarrow \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -i^2 - 1 \\ i - i \end{bmatrix} = \begin{bmatrix} 1-1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 = +i \quad v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\lambda_2 = -i \Rightarrow \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} i \\ -1 \end{bmatrix} = 0 \Rightarrow \quad \lambda_2 = -i \quad v_2 = \begin{bmatrix} i \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \sqrt{1^2 + i^2} = \sqrt{1 + (-1)} = 0$$

$$\begin{bmatrix} 1+i \\ 2-i \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \sqrt{|x|^2 + |y|^2}$$

$$|x| = \qquad x = a + bi \qquad |x| = \sqrt{a^2 + b^2}$$

$$|x|^2 = \underline{\underline{a^2 + b^2}}$$

$$x\bar{x} = (a+bi)(a-bi) = a^2 + \cancel{abi} - \cancel{abi} - b^2 i^2$$
$$= a^2 + b^2 = |x|^2$$

V

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$v \in \mathbb{R}^n \quad \|v\| = \sqrt{v^T v} \Rightarrow \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

$v \in \mathbb{C}^n \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{C}^n \quad \|v\| = \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}$

$= \sqrt{v_1 \bar{v}_1 + v_2 \bar{v}_2 + \cdots + v_n \bar{v}_n}$

$\|v\|^2 = \underset{1 \times n}{[v_1 \ v_2 \cdots v_n]} \underset{n \times 1}{\begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_n \end{bmatrix}} = v^T \bar{v} = \bar{v}^T v$

$[\bar{v}_1 \ \bar{v}_2 \cdots \bar{v}_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \bar{v}^T v$

$A \in \mathbb{R}^{m \times n} \qquad (\bar{A})^T = \bar{A}^T = A^* = A^H = A'$

MATLAB

~~Congu~~ conjugate transpose / Hermitian Transpose

Inner product on $\mathbb{C}$ $\quad \langle \cdot, \cdot \rangle = V \times V \to \mathbb{C}$

$\langle v, v \rangle \geq 0 \qquad \langle v, v \rangle = 0$ only if $\vec{v} = \vec{0}$

bilinear

$\langle u, v \rangle = \overline{\langle v, u \rangle}$ conjugate symmetric

---

$u, v \in \mathbb{C}^n \Rightarrow \langle u, v \rangle = \bar{v}^T u = v^* u = v^H u$ (dot product)

$\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{u^* u}$

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$$\overline{AB} = \bar{A}\,\bar{B}$$

$A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{m\times p}$

$$\overline{a+b} = \bar{a} + \bar{b}$$
$$\overline{ab} = \bar{a}\,\bar{b}$$

$a, b \in \mathbb{C}$

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